CSC 236H 2006, Assignment 3, St. George Campus
Due: Mar. 23, 10:10 AM

On the cover page of your assignment, you must list everyone with whom you discussed this assignment, and which problems you discussed with each person. You must also write **and sign** *the following statement: "I have read and understood the policy on collaboration on homework assignments stated in the Course Information handout." Without these, your homework will not be marked.*

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1. Consider the following recurrence relation: [20]

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2,\ n \in \mathbb{N},\ n \text{ is a power of } 2 \\ T(n-1) + 1 & \text{if } n \geq 2,\ n \in \mathbb{N},\ n \text{ is not a power of } 2 \end{cases}$$

If n is a power of 2, then it is not hard to see that $T(n) = \log n$ as we have had before. However, for this relation, we will show in general that $T(n) \notin \mathcal{O}(\log n)$.

(a) Prove by complete induction that if $n = 2^p + r$, where $p, r \in \mathbb{N}$, and $0 \leq r < 2^p$, then $T(n) = p + r$. (Since every n can be written in this form, this gives us a closed-form solution for T).

(b) Use (a) to prove that if $n = 2^{p+1} - 1$, where $p \in \mathbb{N}, p > 1,$ then $T(n) > n/2$.

(c) It is known that exponentials grow faster than polynomials. In particular, for any constant λ, $2^n > n^\lambda$ provided that n is sufficiently large: $n > 16$ and $n > 2^\lambda$. Use this fact (no proof required) and part (b) to prove that $T(n) \notin \mathcal{O}(\log n)$. *Hint:* One way to do this is as follows. Assume to the contrary that there are constants c and n_0 such that for all $n \geq n_0$, $T(n) \leq c \cdot \log n$. Now choose an n that is sufficiently large and of the form used in part (b) to derive a contradiction. You will want the $\lambda = 2c$.

2. Let $\Sigma = \{(,), 6, 7, 9, +, \times\}$ and define the language $\mathcal{L}$ over the alphabet Σ as the smallest set such that [10]

(a) $6 \in \mathcal{L}$ and $9 \in \mathcal{L}$;

(b) if $u \in \mathcal{L}$ and $v \in \mathcal{L}$, then $(u + v) \in \mathcal{L}$ and $(7 \times v) \in \mathcal{L}$.

Now let val be the function from $\mathcal{L}$ to $\mathbb{N}$ that evaluates the strings in $\mathcal{L}$. For example,

$$\mathsf{val}(\,6\,) = 6$$
$$\mathsf{val}(\,(7 \times 9)\,) = 63$$
$$\mathsf{val}(\,((7 \times (6 + 6)) + (6 + 9))\,) = 99$$

Prove by structural induction that for all $u \in \mathcal{L}$, $\mathsf{val}(u)$ is a multiple of 3.

3. Let $\Sigma = \{(,), 6, 7, 9, +, \times\}$ and define the language $\mathcal{L}'$ over the alphabet Σ as the smallest set such that [10]

   (a) $6 \in \mathcal{L}'$ and $9 \in \mathcal{L}'$;
   (b) if $v \in \mathcal{L}'$, then $(6 + v) \in \mathcal{L}'$, $(9 + v) \in \mathcal{L}'$, and $(7 \times v) \in \mathcal{L}'$.

   If we consider the language $\mathcal{L}$ from the previous question, then clearly $\mathcal{L}' \subseteq \mathcal{L}$ since $\mathcal{L}$ satisfies the two rules for $\mathcal{L}'$. Show that the converse is false by proving that $((6 + 9) + 6) \notin \mathcal{L}'$. *Hint:* Find a property P of strings and prove that $P(u)$ holds for all $u \in \mathcal{L}'$, but that $P(\,(6 + 9) + 6)\,)$ does not hold. Read Example 4.3, p. 101.

4. For each of the following pairs of sentences taken from a first-order language with just two predicates symbols P and Q and no constant symbols, present a structure whose domain is $\{0, 1\}$ where the first sentence is true and the second sentence is false. (No proofs are needed.) [10]

   (a) $\forall x(P(x) \lor Q(x))$ $\quad$ $(\forall x P(x) \lor \forall y P(y))$
   (b) $\forall x(P(x) \lor \neg Q(x))$ $\quad$ $\neg\forall x(P(x) \lor Q(x))$
   (c) $\exists x(P(x) \to Q(x))$ $\quad$ $\exists x(P(x) \land Q(x))$
   (d) $\exists x(P(x) \to \exists y Q(y))$ $\quad$ $(\exists x P(x) \to \exists y Q(y))$
   (e) $\forall y(\forall x P(x) \to Q(y))$ $\quad$ $\forall x(P(x) \to \forall y Q(y))$

5. This is a question about logical implication for a language with no constant symbols and a single binary predicate symbol R. Consider the following three sentences: [20]

   i) $\exists y \forall x R(x, y)$

   ii) $\exists y_1 \exists y_2 \forall x(R(x, y_1) \lor R(x, y_2))$

   iii) $\forall x \exists y R(x, y)$

   (a) Prove that (i) logically implies (ii).
   (b) Prove that (ii) does not logically imply (i).
   (c) Prove that (ii) logically implies (iii).
   (d) Prove that (iii) does not logically imply (ii).