CSC 236H 2006, Assignment 1, St. George Campus
Due: Feb. 9, 10:10 AM

On the cover page of your assignment, you must list everyone with whom you discussed this assignment, and which problems you discussed with each person. You must also write **and sign** *the following statement: "I have read and understood the policy on collaboration on homework assignments stated in the Course Information handout." Without these, your homework will not be marked.*

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1. A group of Toronto and Montreal fans are in line to buy tickets for a game between the Leafs and the Canadiens. The first person in the line is a Toronto fan, and the last in line is a Montreal fan. Prove that somewhere in the line a Toronto fan must be standing directly in front of a Montreal fan (which could be trouble). [10]

   More precisely, prove by simple induction that for all $n \in \mathbb{N}, n \geq 2$, in a line of length n, where person 1 is a Toronto fan, and person n is not a Toronto fan, there exists a k, where $1 \leq k < n$, such that person k is a Toronto fan and person $(k+1)$ is not a Toronto fan.

2. This question has two parts: [10]

   (a) Prove that for all $n \in \mathbb{N}, n \geq 3 :\ n^3 \geq 2n^2 + 2n + 1$. You can prove this using induction, or some other way.

   (b) Using part (a) and induction, prove that for all $n \in \mathbb{N}, n \geq 6 :\ 3^n \geq 3n^3$.

3. Define the sequence of natural numbers, a_0, a_1, a_2, ..., as follows: [10]

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 0 \\ a_2 &= 0 \\ a_n &= a_{n-1} + a_{n-2} + a_{n-3}, \quad \text{for } n \geq 3. \end{aligned}$$

   Although this sequence is a bit slow starting, it does eventually start to grow. Prove by complete induction that $a_n > n^2$ for all integers $n \geq 12$.

4. It is well known that natural numbers can be represented with alternate *bases*. For example, the decimal (base 10) number 46 is written 1201 in ternary (base 3), since $46 = 1{\cdot}3^0 + 0{\cdot}3^1 + 2{\cdot}3^2 + 1{\cdot}3^3$. Note that the coefficients $1, 0, 2, 1$, (from least to most significant) are all drawn from the set $\{0, 1, 2\}$. (For uniqueness, we usually also insist that the most significant coefficient not be zero.) In this question, we consider an *alternate ternary representation*, where the coefficients are drawn from the set $\{-1, 0, 1\}$. In this alternate representation, 46 would be written (1)(-1)(-1)(0)(1), since $46 = 1 \cdot 3^0 + 0 \cdot 3^1 - 1 \cdot 3^2 - 1 \cdot 3^3 + 1 \cdot 3^4$. [10]

Prove using complete induction that every positive natural number has an alternate ternary representation. More precisely, prove that for all $n \in \mathbb{N}, n \geq 1$, there exists a $k \in \mathbb{N}$ and coefficients $c_0, c_1, \ldots, c_k$ where each $c_i \in \{-1, 0, 1\}$ such that

$$n = \sum_{i=0}^{k} c_i \cdot 3^i.$$

Hint: Use the fact that when a natural number n is divided by 3, the remainder is one of 0, 1, or 2. In other words, $n = 3q + r$ where $q \in \mathbb{N}$ and $r \in \{0, 1, 2\}$. However, be careful about the assumptions you make to use the induction hypothesis.

5. Prove that the following program is partially correct for the postcondition $z = m^3$ and precondition $m \in \mathbb{N}$. [10]

```
x := 0
y := 0
z := 0
WHILE x != m DO
   x := x+1
   y := y+2x-1
   z := z+3y-3x+1
END
```

(The program is in fact totally correct, and termination is easy to verify: the program does precisely m iterations.)

6. Consider the following program that calculates cube roots using a binary search: [20]

```
a := 0
b := u
WHILE a != b DO
   g := (a+b) DIV 2
   IF g*g*g < u
     THEN a := g+1
     ELSE b := g
   END
END
```

The precondition here is that u is a cube of a natural number (that is, for some $m \in \mathbb{N}, u = m^3$). The postcondition is that $a^3 = u$, that is, that $a = \sqrt[3]{u}$. (Since $a = b$ at the end, we have that b can be returned as a cube root of u also.)

(a) Prove the program partially correct. *Hint:* prove that "$a \leq \sqrt[3]{u} \leq b$" is a loop invariant.

(b) Prove that the program terminates. *Hint:* use $E_n = (b_n - a_n)$ and the analysis of *BinSearch* in the book, p. 49-54, especially, Lemma 2.2 (for DIV).

(c) Suppose the `THEN` clause of the program had been "`THEN a := g`." Would the program still work? Either explain informally why it is still correct, or show it behaving incorrectly for some value of u.

(d) Suppose the `WHILE` clause of the program had been "`WHILE a*a*a != u DO`." Would the program still work? Either explain informally why it is still correct, or show it behaving incorrectly for some value of u.