QUESTION 1. [5 MARKS]

Define $f(n)$ as:

$$f(n) = \sum_{i=0}^{2n+1} 7^i.$$

Prove that $f(n)$ is divisible by 8 for all $n \in \mathbb{N}$.

SOLUTION: This question closely resembles Assignment 1, Q6a. The only difference is the base here is 7, rather than 2.

CLAIM: $P(n)$: "$f(n)$ is divisible by 8" is true for all $n \in \mathbb{N}$

PROOF (SIMPLE INDUCTION ON n): $P(0)$ asserts that $\sum_{i=0}^{1} 7^i$ is divisible by 8, or $7^0 + 7^1 = 8$ is divisible by 8. This is certainly true, since $8 = 8 \times 1 + 0$, so the base case holds.

INDUCTION STEP: I wish to show that for any n, $P(n) \implies P(n+1)$, so I assume $P(n)$ for an arbitrary $n \in \mathbb{N}$, in other words I assume that $\sum_{i=0}^{2n+1} 7^i = 8k$, for some $k \in \mathbb{N}$ (this is the IH). Now I can break up the sum $\sum_{i=0}^{2(n+1)+1}$ and use the induction hypothesis:

$$\begin{aligned}
\sum_{i=0}^{2(n+1)+1} 7^i &= 7^0 + 7^1 + \left(\sum_{i=2}^{2(n+1)+1} 7^i \right) \\
\text{[factor out } 7^2\text{]} &= 8 + \left(7^2 \sum_{i=0}^{2n+1} 7^i \right) \\
\text{[by IH]} &= 8 + \left(7^2 8k \right) = 8(1 + 7^2 k).
\end{aligned}$$

Thus $\sum_{i=0}^{2(n+1)+1} 7^i$ is divisible by 8, so $P(n) \implies P(n+1)$, as wanted.

I conclude that $P(n)$ holds for all $n \in \mathbb{N}$. QED.

STATE AND VERIFY BASIS: 1 mark. -0.5 if $f(n)$ is used as a predicate. -1 for making $\forall n$ part of the predicate. -0.5 if the base case omits $n = 0$ (and starts at $n = 1$).

SET UP INDUCTION: 1 mark. You need to state something equivalent to "I will show that $P(n)$ implies $P(n+1)$," or "assume $P(n)$ for an arbitrary $n \in \mathbb{N}$, now show $P(n+1)$."

INDUCTION STEP: 2 marks. Show that $P(n) \Rightarrow P(n+1)$. -1 if step where IH is used is not explicitly shown.

CONCLUSION: 1 mark. Conclude that $P(n)$ is true for all n.

QUESTION 2. [5 MARKS]

For $n \in \mathbb{N}$, define $B(n)$ as:

$$B(n) = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ B(n-2) + B(n-1), & n > 1 \end{cases}.$$

Prove that $B(n+2) - \sum_{i=0}^{n} B(i) = 1$ for all $n \in \mathbb{N}$.

STATE AND VERIFY BASE CASE: 1 mark. -0.5 if you don't state what claim your algebra is verifying.

SET UP INDUCTION: 1 mark. Either assume $P(n)$ for some arbitrary n, or say you will show that $P(n) \Rightarrow P(n+1)$. -1 mark for assuming $P(n)$ for all n.

INDUCTION STEP: Show that $P(n) \Rightarrow P(n+1)$. -1 mark if you don't indicate where IH is used. -1 mark if you don't indicate where definition of U is used.

CONCLUSION: Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

SOLUTION: This question closely resembles Assignment 2, Q2a. The difference is that the recursively-defined function has different starting conditions.

CLAIM: $P(n)$: "$B(n) - \sum_{i=0}^{n} B(i) = 1$" is true for all $n \in \mathbb{N}$.

PROOF (SIMPLE INDUCTION ON n): $P(0)$ asserts that $B(2) - \sum_{i=0}^{0} B(i) = 1$, or in other words $2 - 1 = 1$, which is certainly true, so the base case holds.

INDUCTION STEP: In order to prove that for any $n \in \mathbb{N}$, $P(n) \implies P(n+1)$, I assume $P(n)$ for an arbitrary $n \in \mathbb{N}$. In other words, my induction hypothesis (IH) is that $B(n+2) - \sum_{i=0}^{n} B(i) = 1$. Now I can re-write $B(n+1+2) - \sum_{i=0}^{n+1} B(i)$, and use the induction hypothesis

$$\begin{aligned} B(n+1+2) - \sum_{i=0}^{n+1} B(i) &= B(n+3) - \left(\sum_{i=0}^{n} B(i)\right) - B(n+1) \\ \text{[by IH]} &= B(n+3) - (B(n+2) - 1) - B(n+1) \\ \text{[by definition of } B(n+3)\text{]} &= B(n+3) - B(n+3) + 1 = 1. \end{aligned}$$

Thus $P(n)$ implies $P(n+1)$, as wanted.

I conclude that $P(n)$ holds for all $n \in \mathbb{N}$. QED.

STATE AND VERIFY BASE CASE: 1 mark. -0.5 if you don't state what claim your algebra is verifying.

SET UP INDUCTION: 1 mark. Either assume $P(n)$ for some arbitrary n, or say you will show that $P(n) \Rightarrow P(n+1)$. -1 mark for assuming $P(n)$ for all n.

INDUCTION STEP: 2 marks. Show that $P(n) \Rightarrow P(n+1)$. -1 mark if you don't indicate where IH is used. -1 mark if you don't indicate where definition of U is used.

CONCLUSION: 1 mark. Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

QUESTION 3. [5 MARKS]

Let $PV = \{v, w, x, y, z\}$ be a set of propositional variables. Define a special set of propositional formulas $\mathcal{F}^*$ as the smallest set such that

BASIS: Any propositional variable in PV belongs to $\mathcal{F}^*$.

INDUCTION STEP: If P_1 and P_2 belong to $\mathcal{F}^*$, then so do $(P_1 \wedge P_2)$, $(P_1 \vee P_2)$, $(P_1 \rightarrow P_2)$ and $(P_1 \leftrightarrow P_2)$.

For a propositional formula f, define $\mathbf{cn}(f)$ as the number of instances of connectives from $\{\vee, \wedge, \rightarrow, \leftrightarrow\}$ in f. Define $\mathbf{pv}(f)$ as the number of instances of propositional variables from $\{v, w, x, y, z\}$ in f.

Use structural induction to prove that for all $f \in \mathcal{F}^*$, $\mathbf{pv}(f) = \mathbf{cn}(f) + 1$.

SOLUTION: This question resembles the example worked in lecture (see lecture summary for Week 6).

CLAIM: $P(f)$: "$\mathbf{pv}(f) = \mathbf{cn}(f) + 1$" is true for all $f \in \mathcal{F}^*$.

PROOF (STRUCTURAL INDUCTION ON f:): For the basis, it is enough to check $f = u$, $f = v$, $f = x$, $f = y$, and $f = z$. In each case there is a single propositional variable and no connectives, so $\mathbf{pv}(f) = 1 = \mathbf{cn}(f) + 1$. Thus the base case holds.

INDUCTION STEP: Assume that $P(f_1)$ and $P(f_2)$ both holds, and that $f = (f_1 * f_2)$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Notice that in each case, f has the same number of propositional variables, and one more connective, than f_1 and f_2 do combined, so the following observations (observation 1 on the left, observation 2 on the right) hold:

$$\mathbf{pv}(f) = \mathbf{pv}(f_1) + \mathbf{pv}(f_2) \qquad \qquad \mathbf{cn}(f) = \mathbf{cn}(f_1) + \mathbf{cn}(f_2) + 1.$$

You can now combine these two observations to show:

$$\begin{aligned}
\text{[by observation 1]} \quad \mathbf{pv}(f) &= \mathbf{pv}(f_1) + \mathbf{pv}(f_2) \\
\text{[by IH for } f_1 \text{ and } f_2\text{]} &= \mathbf{cn}(f_1) + 1 + \mathbf{cn}(f_2) + 1 \\
\text{[by commutativity of addition]} &= \mathbf{cn}(f_1) + \mathbf{cn}(f_2) + 1 + 1 \\
\text{[by observation 2]} &= \mathbf{cn}(f) + 1.
\end{aligned}$$

This is exactly what $P(f)$ asserts, so $P(f_1)$ and $P(f_2)$ imply $P(f)$, as wanted.

I conclude that $P(f)$ holds for all $f \in \mathcal{F}^*$. QED.

STATE AND VERIFY BASIS: 0.5 marks. Check that $P(f)$ holds when f is a propositional variable.

SET UP INDUCTION: 1 mark. Show the connection between a new formula and formulas about which the property, P, is assumed.

INDUCTION STEP: 3 marks. Show that $P(f_1)$ and $P(f_2)$ imply $P(f)$. -1 for not indicating where IH is used. -0.5 if observations about number of connectives, parentheses, or variables in f versus those in subformulas are not explained.

CONCLUSION: 0.5 marks. Conclude that property P holds for all $f \in \mathcal{F}^*$.

REMARKS: There were some attempts to use simple induction on $pv(f)$ (this won't work). There were some incorrect basis cases (not propositional variables).

Total Marks = 15