QUESTION 1. [5 MARKS]

Define $g(n)$ as:

$$g(n) = \sum_{i=0}^{2n} 5^i.$$

Prove that $g(n)$ mod 6 is equal to 1 for all $n \in \mathbb{N}$. In other words, prove that for each $n \in \mathbb{N}$, there is some integer k such that $g(n) = 6k + 1$.

SOLUTION: This question closely resembles Assignment 1, Q6. The difference is that here the base is 5, rather than 2.

CLAIM: $P(n)$: "$g(n)$ mod 6 is equal to 1" is true for all $n \in \mathbb{N}$.

PROOF (SIMPLE INDUCTION ON n): $P(0)$ asserts that $\left(\sum_{i=0}^{0} 5^i\right)$ mod 6 is equal to 1. In other words, 1 mod 6 = 1, which is certainly true since $1 = 6 \times 0 + 1$. Thus the base case holds.

INDUCTION STEP: In order to prove that $P(n)$ implies $P(n+1)$ for an arbitrary natural number n, I assume $P(n)$. In other words, my induction hypothesis is that $\left(\sum_{i=0}^{2n} 5^i\right) = 6k + 1$, for some integer k. Now I can re-write $g(n+1) = \sum_{i=0}^{2(n+1)}$ and use the IH:

$$\begin{aligned}
\sum_{i=0}^{2(n+1)} 5^i &= 5^0 + 5^1 + \left(\sum_{i=2}^{2(n+1)} 5^i\right) \\
\text{[factoring out } 5^2\text{]} &= 6 + \left(5^2 \sum_{i=0}^{2n} 5^i\right) \\
\text{[by IH]} &= 6 + 5^2 6k + 1 = 6(1 + 5^2 k) + 1.
\end{aligned}$$

So $g(n+1)$ mod 6 equals 1, and $P(n) \Rightarrow P(n+1)$, as wanted.

I conclude that $P(n)$ holds for all $n \in \mathbb{N}$. QED.

STATE AND VERIFY BASIS: 1 mark. -0.5 if $g(n)$ is used as a predicate. -1 for making $\forall n$ part of the predicate. -0.5 if the base case omits $n = 0$ (and starts at $n = 1$).

SET UP INDUCTION: 1 mark. You need to state something equivalent to "I will show that $P(n)$ implies $P(n+1)$," or "assume $P(n)$ for an arbitrary $n \in \mathbb{N}$, now show $P(n+1)$."

INDUCTION STEP: 2 marks. Show that $P(n) \Rightarrow P(n+1)$. -1 if step where IH is used is not explicitly shown.

CONCLUSION: 1 mark. Conclude that $P(n)$ is true for all n.

REMARKS: There is no need to use induction to prove that $5^n + 5^{n+1}$ is divisible by 6 — algebra will do. Also, many answers omitted a term from $\sum_{i=0}^{2m+2} 5^i = (\sum_{i=0}^{2m} 5^i) + 5^{2m+1} + 5^{2m+2}$.

QUESTION 2. [5 MARKS]

For $k \in \mathbb{N}$, define $U(3^k)$ as:

$$U(3^k) = \begin{cases} c, & k = 0 \\ 3U(3^{k-1}) + d3^k, & k > 0 \end{cases}.$$

Prove that for all $k \in \mathbb{N}$, $U(3^k) = 3^k(c + dk)$.

SOLUTION: This question closely resembles the exercise worked in lecture for the complexity of `MergeSort`, in the on-line lecture summary for Week 4.

CLAIM: $P(k)$ "$U(3^k) = 3^k(c + dk)$" is true for all $k \in \mathbb{N}$.

PROOF (SIMPLE INDUCTION ON k): $P(0)$ asserts that $U(1) = c$, which is true by the definition of $U(3^0)$. Thus the base case holds.

INDUCTION STEP: I want to show that for any $k \in \mathbb{N}$, $P(k) \Rightarrow P(k+1)$, so assume that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$. In other words, I assume that $U(3^k) = 3^k(c + dk)$. Now I can unwind $U(3^{k+1})$ and apply this induction hypothesis

$$\begin{aligned} \text{[by definition of } U(3^{k+1})] \qquad U(3^{k+1}) &= 3U(3^k) + d3^{k+1} \\ \text{[by IH]} \quad &= 3\left(3^k(c + dk)\right) + d3^{k+1} \\ &= 3^{k+1}c + 3^{k+1}dk + 3^{k+1}d = 3^{k+1}(c + d[k+1]). \end{aligned}$$

This is exactly what $P(k+1)$ asserts, so $P(k) \Rightarrow P(k+1)$, as wanted.

I conclude that $P(k)$ is true for all $k \in \mathbb{N}$.

STATE AND VERIFY BASE CASE: 1 mark. -0.5 if you don't state what claim your algebra is verifying.

SET UP INDUCTION: 1 mark. Either assume $P(n)$ for some arbitrary n, or say you will show that $P(n) \Rightarrow P(n+1)$. -1 mark for assuming $P(n)$ for all n.

INDUCTION STEP: 2 marks. Show that $P(n) \Rightarrow P(n+1)$. -1 mark if you don't indicate where IH is used. -1 mark if you don't indicate where definition of U is used.

CONCLUSION: 1 mark. Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

QUESTION 3. [5 MARKS]

Let $PV = \{v, w, x, y, z\}$ be a set of propositional variables. Define a special set of propositional formulas $\mathcal{F}^*$ as the smallest set such that

BASIS: Any propositional variable in PV belongs to $\mathcal{F}^*$.

INDUCTION STEP: If P_1 and P_2 belong to $\mathcal{F}^*$, then so do $(P_1 \wedge P_2)$, $(P_1 \vee P_2)$, $(P_1 \rightarrow P_2)$ and $(P_1 \leftrightarrow P_2)$.

For a propositional formula f, define $\mathbf{cn}(f)$ as the number of instances of connectives from $\{\vee, \wedge, \rightarrow, \leftrightarrow\}$ in f. Define $\mathbf{p}(f)$ as the number of parentheses in f.

Use structural induction to prove that for all $f \in \mathcal{F}^*$, $\mathbf{p}(f) = 2\mathbf{cn}(f)$.

SOLUTION: This question resembles the example worked in lecture (see lecture summary for Week 6).

CLAIM: $P(f)$: "$\mathbf{p}(f) = 2\mathbf{cn}(f)$" is true for all $f \in \mathcal{F}^*$.

PROOF (STRUCTURAL INDUCTION ON f): For the basis it is enough to verify that $P(f)$ holds for $f = v$, $f = w$, $f = x$, $f = y$, and $f = z$. In each case there are no connectives or parentheses in f, so $\mathbf{p}(f) = 2\mathbf{cn}(f) = 0$. Thus the base case holds.

INDUCTION STEP: Assume that $P(f_1)$ and $P(f_2)$ hold, and that $f = (f_1 * f_2)$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Notice that in each case, f has two more parentheses and one more connective than f_1 and f_2 have combined, so the following two observations (observation 1 on the left, observation 2 on the right) hold:

$$\mathbf{p}(f) = \mathbf{p}(f_1) + \mathbf{p}(f_2) + 2 \qquad \mathbf{cn}(f) = \mathbf{cn}(f_1) + \mathbf{cn}(f_2) + 1.$$

Using these two observations, you can see that

$$\begin{aligned} \text{[by observation 1]} \quad \mathbf{p}(f) &= \mathbf{p}(f_1) + \mathbf{p}(f_2) + 2 \\ \text{[by IH for } f_1 \text{ and } f_2\text{]} &= 2\mathbf{cn}(f_1) + 2\mathbf{cn}(f_2) + 2 = 2(\mathbf{cn}(f_1) + \mathbf{cn}(f_2) + 1) \\ \text{[by observation 2]} &= 2\mathbf{cn}(f). \end{aligned}$$

Thus $P(f_1)$ and $P(f_2)$ imply $P(f)$, as wanted.
I conclude that $P(f)$ holds for all $f \in \mathcal{F}^*$. QED.

STATE AND VERIFY BASIS: 0.5 marks. Check that $P(f)$ holds when f is a propositional variable.

SET UP INDUCTION: 1 mark. Show the connection between a new formula and formulas about which the property, P, is assumed.

INDUCTION STEP: 3 marks. Show that $P(f_1)$ and $P(f_2)$ imply $P(f)$. -1 for not indicating where IH is used. -0.5 if observations about number of connectives, parentheses, or variables in f versus those in subformulas are not explained.

CONCLUSION: 0.5 marks. Conclude that property P holds for all $f \in \mathcal{F}^*$.

REMARKS: There were some attempts to use simple induction on $pv(f)$ (this won't work). There were some incorrect basis cases (not propositional variables).

Total Marks = 15