Do not turn this page until you have received the signal to start.
(In the meantime, please fill out the identification section above, and read the instructions below.)

This exam consists of 3 questions on 4 pages (including this one) When you receive the signal to start, please make sure that your copy of the examination is complete.

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write “I cannot answer this question,” on. You will earn substantial part marks for writing down the outline of a proof and indicating which steps are missing.

Write your student number at the bottom of pages 2-4 of this test.

# 1: _____/ 5
# 2: _____/ 5
# 3: _____/ 5

TOTAL: _____/ 15

Good Luck!
QUESTION 1. [5 MARKS]

Define $g(n)$ as:

$$g(n) = \sum_{i=0}^{2n} 5^i.$$

Prove that $g(n)$ mod 6 is equal to 1 for all $n \in \mathbb{N}$. In other words, prove that for each $n \in \mathbb{N}$, there is some integer $k$ such that $g(n) = 6k + 1$. 
QUESTION 2.  [5 MARKS]

For $k \in \mathbb{N}$, define $U(3^k)$ as:

$$U(3^k) = \begin{cases} 
  c, & k = 0 \\
  3U(3^{k-1}) + d3^k, & k > 0
\end{cases}.$$

Prove that for all $k \in \mathbb{N}$, $U(3^k) = 3^k(c + dk)$. 
QUESTION 3  [5 marks]

Let \( PV = \{v, w, x, y, z\} \) be a set of propositional variables. Define a special set of propositional formulas \( \mathcal{F}^* \) as the smallest set such that

**Basis**: Any propositional variable in \( PV \) belongs to \( \mathcal{F}^* \).

**Induction Step**: If \( P_1 \) and \( P_2 \) belong to \( \mathcal{F}^* \), then so do \( (P_1 \land P_2) \), \( (P_1 \lor P_2) \), \( (P_1 \rightarrow P_2) \) and \( (P_1 \leftrightarrow P_2) \).

For a propositional formula \( f \), define \( \text{cn}(f) \) as the number of instances of connectives from \( \{\lor, \land, \rightarrow, \leftrightarrow\} \) in \( f \). Define \( p(f) \) as the number of parentheses in \( f \).

Use structural induction to prove that for all \( f \in \mathcal{F}^* \), \( p(f) = 2\text{cn}(f) \).

Total Marks = 15