QUESTION 1. [10 MARKS]

Prove that if the precondition is true when `zigzag(int n)` starts, then `zigzag(int n)` terminates.

```
/*
 * Precondition: n is an integer
 */
public static void zigzag(int n) {
  int i = 1;
  if (n < 0) {
    i = -1;
  }

  while (n != 0) {
    n = n - i;
    i = i * -1;
    n = n * -1;
  }
}
```

CLAIM 1A: Let $P(j)$ be "If there is a jth iteration of the loop, then if $n_j > 0$ $i_j = 1$ and if $n_j < 0$, $i_j = -1$." Then for all $j \in \mathbb{N}, P(j)$.

PROOF (INDUCTION ON j): If $j = 0$ then i_j is set to 1 by the first line. If $n_0 > 0$, then the "if $(n < 0)$" branch is not executed, and both $n_j > 0$ and $i_j = 1$. Otherwise the "if $(n < 0)$" branch is executed, and $n_j < 0$ and $i_j = -1$. Thus the claim holds for $j = 0$.

INDUCTION STEP: Assume that $P(j)$ is true for some arbitrary natural number j. I want to show that this implies $P(j+1)$. If there is no $(j+1)$th iteration of the loop, $P(j)$ is vacuously true. Otherwise $n_j \neq 0$, in order for the loop condition to be satisfied. There are two cases to consider

$n_j > 0$: By $P(j)$ this means that $i_j = 1$. Thus, examining the while loop, $i_{j+1} = -1$ and $n_{j+1} = (-1)(n_j - 1) = 1 - n_j$. If $n_j = 1$, then $n_{j+1} = 0$ and there is nothing prove (an empty antecedent in $P(j+1)$). Otherwise $n_j > 1$, so $n_{j+1} < 0$, and $P(j+1)$ holds.

$n_j < 0$: By $P(j)$ this means that $i_j = -1$. Thus, examining the while loop, $i_{j+1} = 1$ and $n_{j+1} = (-1)(n_j + 1) = (-n_j - 1)$. If $n_j = -1$ then $n_{j+1} = 0$, and there is nothing prove (an empty antecedent in $P(j+1)$. Otherwise, $n_j < -1$, so $n_{j+1} > 0$ and $P(j+1)$ holds.

In both case $P(j+1)$ holds, so $P(j) \Rightarrow P(j+1)$.

I conclude that $P(j)$ is true for all $j \in \mathbb{N}$. QED.

CLAIM 1B: If there is a $(j+1)$th iteration of the loop, then $|n_j| > |n_{j+1}|$.

PROOF: If there is a $(j+1)$th iteration of the loop then $n_j \neq 0$ (to satisfy the loop condition), so $|n_j| > 0$. Thus if $n_{j+1} = 0$, we're done and $|n_j| > |n_{j+1}|$. Otherwise, either $n_j > 0$ and (by $P(j)$) $i_j = 1$, so $|n_{j+1}| = |1 - n_j| < |n_j|$, or else $n_j < 0$, and (by $P(j)$ again) $|n_{j+1}| = |(-1)(n_j + 1)| = |n_j + 1| < |n_j|$. Thus, in each case, $|n_j| > |n_{j+1}|$. QED.

CLAIM 1C: `zigzag(int n)` terminates.

PROOF: Since n is of type int, $|n_j|$ is a non-negative integer, i.e. a natural number. Furthermore, we have shown that if there is a $(j+1)$th iteration, $|n_j| > |n_{j+1}|$. This means that the sequence $\langle |n_j| \rangle$ is a strictly decreasing sequence of natural numbers, and non-empty since it contains at least $|n_0|$. There is thus (PWO) a least value, say $|n_k|$, which must also be last (strictly decreasing), which means there is no $(k+1)$th iteration of the loop — it terminates. QED.

MARKING SCHEME: Six marks for showing that there is a strictly-decreasing sequence of natural numbers associated with the iterations of the loop, four marks for showing this sequence is non-empty and using the PWO to show that this implies that the sequence is finite, and hence the loop terminates.

As part of the six marks, you may need to show that $|i_j| = 1$, that i_j and n_j have the same sign (except deal with the case where $n_j = 0$), and deal with the cases where $n_j > 0$ and $n_j < 0$.

As part of the four marks, you'll have to appeal to there being at least one element (say $|n_0|$) in the sequence, that the sequence corresponds to a non-empty subset of $\mathbb{N}$, that such a subset has a least member, and that (due to the sequence being decreasing) such a member must be the last, so the sequence (and the associated loop iterations) is finite.

You don't get marks for saying that the PWO implies that n is eventually zero (because the PWO doesn't say this), or for arguing (or describing) how the absolute value gets smaller and smaller until it is eventually zero.

QUESTION 2. [10 MARKS]

Write a formula in Prenex Normal Form (PNF) that is logically equivalent (LEQV) to:

$$((\forall x P(x) \to \forall x Q(x)) \vee (\neg\forall x P(x) \to \neg\forall x Q(x)))$$

Is the formula valid, satisfiable, or unsatisfiable? Briefly justify your answer. (Please note that the last page of this test lists some of the possibly useful logical equivalences for first-order formulas).

$$
\begin{array}{rll}
 & ((\forall x P(x) \to \forall x Q(x)) \vee (\neg\forall x P(x) \to \neg\forall x Q(x))) & \text{LEQV} \\
\text{[re-naming]} & ((\forall x P(x) \to \forall w Q(w)) \vee (\neg\forall v P(v) \to \neg\forall u Q(u))) & \text{LEQV} \\
\text{[re-naming, negating, factoring]} & (\exists x \forall w (P(x) \to Q(w)) \vee (\exists v \neg P(v) \to \exists u \neg Q(u))) & \text{LEQV} \\
\text{[factoring over } \to\text{]} & (\exists x \forall w (P(x) \to Q(w)) \vee \exists u \forall v (\neg P(v) \to \neg Q(u))) & \text{LEQV} \\
\text{[factoring over } \vee\text{]} & \exists x \forall w \exists u \forall v ((P(x) \to Q(w)) \vee (\neg P(v) \to \neg Q(u))) &
\end{array}
$$

The formula is valid, it is satisfied by every interpretation. Let $\forall x P(x) = P'$ (a proposition, since it has no free variables), and $\forall x Q(x) = Q'$ (another proposition, since it has no free variables). Then the formula is equivalent to:

$$
\begin{array}{llll}
(P' \to Q') \vee (\neg P' \to \neg Q') & \text{LEQV} & (\neg P' \vee Q') \vee (P' \vee \neg Q') & [\to \text{ law}] \\
 & \text{LEQV} & (P' \vee \neg P') \vee (Q' \vee \neg Q') & \text{[commutativity, associativity, a tautology]}
\end{array}
$$

MARKING SCHEME: Six marks for showing the derivation of an equivalent PNF formula. One mark deducted for not citing the rules used (from the appendix at the back of the test), two marks deducted for factoring quantifiers over formulas that have the quantified variable as a free variable, two marks deducted for the result not being PNF.

Two marks for saying the formula is valid, and two marks for justifying this conclusion. One mark if you say the formula is satisfiable but don't say it's valid, plus (possibly) another mark for justifying this partial conclusion.

QUESTION 3. [10 MARKS]

Which of the following claims are true, which are false? Justify your answers using truth assignments, truth tables or the equivalances from lecture (double negation, De Morgan's laws, commutative laws, associative laws, distributive laws, identity laws, idempotency laws, $\rightarrow$ law, $\leftrightarrow$ law).

PART (A) [4 MARKS]

$(x \wedge y) \rightarrow z$ LEQV $(x \rightarrow z) \vee (y \rightarrow z)$.

The claim is true, as shown by the following equivalences:

$$\begin{aligned}
(x \wedge y) \rightarrow z \quad &\text{LEQV} \quad \neg(x \wedge y) \vee z && [\rightarrow \text{rule}] \\
&\text{LEQV} \quad (\neg x \vee \neg y) \vee z && [\text{De Morgan's law}] \\
&\text{LEQV} \quad (\neg x \vee \neg y) \vee z \vee z && [\text{idempotency}] \\
&\text{LEQV} \quad \neg x \vee (\neg y \vee z) \vee z && [\text{Associativity}] \\
&\text{LEQV} \quad (\neg y \vee z) \vee (\neg x \vee z) && [\text{commutativity, associativity}] \\
&\text{LEQV} \quad (y \rightarrow z) \vee (x \rightarrow z) && [\rightarrow \text{law}]
\end{aligned}$$

PART (B) [3 MARKS]

$x \leftrightarrow (y \leftrightarrow z)$ LEQV $(x \leftrightarrow z) \leftrightarrow y$.

The claim is true, since the formulas are satisfied by exactly the same truth assignments in the following truth table:

x	y	z	$(y \leftrightarrow z)$	$(x \leftrightarrow z)$	$x \leftrightarrow (y \leftrightarrow z)$	$(x \leftrightarrow z) \leftrightarrow y$
0	0	0	1	1	0	0
0	0	1	0	0	1	1
0	1	0	0	1	1	1
0	1	1	1	0	0	0
1	0	0	1	0	1	1
1	0	1	0	1	0	0
1	1	0	0	0	0	0
1	1	1	1	1	1	1

PART (C) [3 MARKS]

$(x \vee y) \rightarrow z$ LEQV $(x \rightarrow z) \wedge (y \rightarrow z)$.

The claim is true, as shown by the following equivalences:

$$\begin{aligned}(x \vee y) \rightarrow z \quad & \text{LEQV} \quad \neg(x \vee y) \vee z && [\rightarrow \text{ law}]\\ & \text{LEQV} \quad (\neg x \wedge \neg y) \vee z && [\text{De Morgan's law}]\\ & \text{LEQV} \quad (\neg x \vee z) \wedge (\neg y \vee z) && [\text{distributivity}]\\ & \text{LEQV} \quad (x \rightarrow z) \wedge (y \rightarrow z) && [\rightarrow \text{ law}]\end{aligned}$$

MARKING SCHEME: One mark deducted for not saying whether you believe the claims are true or false (only deducted once). One mark deducted (only once) for not citing the rules used. One mark awarded for claiming that a true claim is false. One mark awarded for unclear progress towards a conclusion.

Total Marks = 30