QUESTION 1. [5 MARKS]

A binary string is a (possibly empty) sequence of 0's and 1's. Let $B(n)$ be the number of binary strings of length n. Use simple induction to prove that for all $n \in \mathbb{N}$, $B(n) = 2^n$.

Let $P(n)$ be "$B(n) = 2^n$."

CLAIM $P(n)$ is true for all $n \in \mathbb{N}$.

PROOF (INDUCTION ON n) : If $n = 0$, then $P(0)$ asserts that $B(n) = 2^0 = 1$ binary string of length 0. This is certainly true, since the unique binary string of length 0 is the empty string.

INDUCTION STEP: Assume that $P(n)$ is true for some arbitrary natural number n. I want to show that this implies $P(n+1)$. I can partition the binary strings of length $n+1$ into two sets, those that end in the digit 1, and those that end in the digit 0. Binary strings of length $n+1$ ending with the digit 1 consist of an arbitrary binary string of length n, with the suffix 1, so by the IH there are $B(n) = 2^n$ of them. Similarly, binary strings of length $n+1$ that end in 0 consist of an arbitrary binary string of length n with a suffix 0, so by the IH there are $B(n) = 2^n$ of them. This accounts for all binary strings of length $n+1$, and counts none of them twice (since they end with either a 1 or a 0, and not both), so there are $2^n + 2^n = 2^{n+1}$ binary strings of length $n+1$, so $B(n+1) = 2^{n+1}$. This is exactly what $P(n+1)$ claims, so $P(n) \Rightarrow P(n+1)$.

I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

MARKING SCHEME: -1 mark for confusing the function $B(n)$ (which has a numerical value) with some predicate $B(n)$ (a predicate has a boolean value). -1 mark for including "$\forall n$" in the predicate to be proved or the induction hypothesis. -1 mark for assuming $P(n)$ and then "proving" it. -0.5 marks for a small gap or error in your argument, or a confused argument. -2 marks if there is no explanation of why the induction step works. -1 mark for a missing conclusion. -2 marks if the induction step argues about a power set (set of subsets) rather than binary strings, without showing that they are connected.

QUESTION 2. [5 MARKS]

Define $f(n)$ by

$$f(n) = \begin{cases} 5, & n = 0 \\ 5, & n = 1 \\ f(n-1) + 6f(n-2), & n > 1 \end{cases}$$

Let $P(n)$ be "$f(n) = 3^{n+1} + 2(-2)^n$." Prove that $P(n)$ is true for all $n \in \mathbb{N}$.

CLAIM: $P(n)$ is true for all $n \in \mathbb{N}$.

PROOF (COMPLETE INDUCTION ON n): When $n = 0$, $P(0)$ asserts that $f(0) = 3^1 + 2(-2)^0 = 5$, so $P(0)$ is true, by inspecting the definition of $f(0)$. When $n = 1$, $P(1)$ asserts that $f(1) = 3^2 + 2(-2)^1 = 5$, so $P(1)$ is true, by inspecting the definition of $f(1)$. Thus the base cases hold.

INDUCTION STEP: Assume that $P(\{0, \ldots, n-1\})$ is true, for some arbitrary natural number n. I want to show that this implies $P(n)$. If $n \leq 1$, then $P(n)$ was established in the base cases. Otherwise, if $n > 1$ then $0 \leq n-1, n-2 < n$, so we have assumed $P(n-1)$ and $P(n-2)$ in the IH, and

$$\begin{aligned}
f(n) &= f(n-1) + 6f(n-2) && \text{[by definition of } f(n),\ n > 1] \\
&= 3^n + 2(-2)^{n-1} + 6(3^{n-1} + 2(-2)^{n-2}) && \text{[by IH } P(n-1) \text{ and } P(n-2)] \\
&= 3^n + 2(-2)^{n-1} + 2(3)^n + 3(4(-2)^{n-2}) \\
&= 3^{n+1} - (-2)^n + 3(-2)^n && [4 = (-2)^2] \\
&= 3^{n+1} + 2(-2)^n.
\end{aligned}$$

This is exactly what $P(n)$ claims, so $P(\{0, \ldots, n-1\}) \Rightarrow P(n)$.

I conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

MARKING SCHEME: -1 mark for incorrect base cases. -1 mark for missing induction hypothesis, or assuming $P(n)$ for $n > 1$ -1 mark for not setting up complete induction properly, so that the IH cover $n-1$ and $n-2$. Between -0.5 and -1 marks for missing algebra. -1 for solving the related quadratic equation, without proving that the closed form is equal to $f(n)$. -1 mark for assuming $\forall n P(n)$.

QUESTION 3. [5 MARKS]

Let $\mathcal{E}$ be defined as the smallest set such that:

BASIS: x, y, and z are elements of $\mathcal{E}$.

INDUCTION STEP: If e_1 and e_2 are elements of $\mathcal{E}$, then so is (e_1, e_2).

Let $lp(e)$ be the number of left parentheses in e, and $c(e)$ be the number of commas in e. Prove there is no element $e \in \mathcal{E}$ such that $lp(e) = c(e) + 19$. (The left parenthesis is the "(" character, and the comma is the "," character).

CLAIM: Define $lp(e)$ as the number of left parentheses in e, and $c(e)$ as the number of commas in e, and let $P(e)$ be "$lp(e) = c(e)$." Then $\forall e \in \mathcal{E}, P(e)$.

PROOF (STRUCTURAL INDUCTION ON e): If e is defined in the basis, then $e \in \{x, y, z\}$,and so $lp(e) = 0 = c(e)$, since these expressions have no left parentheses nor commas. Thus $P(e)$ is true for the basis.

INDUCTION STEP: Let e_1 and e_2 be arbitrary elements of $\mathcal{E}$, assume that $P(e_1)$ $P(e_2)$ hold, and $e = (e_1, e_2)$. Then e has one more comma than the sum of the commas in e_1 and e_2 (it adds one in the middle), and one more left parenthesis than the sum of the left parentheses in e_1 and e_2 (it adds one on the left), so

$$\begin{aligned} lp(e) &= lp(e_1) + lp(e_2) + 1 \qquad \text{[by the remark above]} \\ &= c(e_1) + c(e_2) + 1 \qquad \text{[by } P(e_1) \text{ and } P(e_2)\text{]} \\ &= c(e) \qquad \text{[by the remark above]} \end{aligned}$$

This is exactly what $P(e)$ asserts, so $P(\{e_1, e_2\}) \Rightarrow P(e)$.

I conclude that $P(e)$ is true for all $e \in \mathcal{E}$. QED.

CLAIM: No element of $\mathcal{E}$ has 19 more left parentheses than it has commas.

PROOF: By $P(e)$ (above), every element of $\mathcal{E}$ has the same number of left parentheses as commas, hence no element has 19 more left parentheses than commas. QED.

MARKING SCHEME: -1 mark for messing up the construction of $\mathcal{E}$ so that extraneous elements are included. -1 mark for omitting any explanation of why $lp(e) = lp(e_1) + lp(e_2) + 1$, or why $c(e) = c(e_1) + c(e_2) + 1$. -1 mark for confusing a predicate, say $P(e)$ (boolean valued), with an expression in $\mathcal{E}$. -1 mark for omitting the induction hypothesis. -2 for no explanation of why $lp(e) = c(e)$. -1 for assuming $P(e), \forall e \in \mathcal{E}$ before proving it.

Total Marks = 15