CSC236 QUIZ 7, TUESDAY JULY 12

Name: Student number:

Prove or disprove the following (no induction needed).

1. If $n > 12$, then $1 \leq \lfloor n/11 \rfloor \leq \lceil n/11 \rceil < n$.

   SAMPLE SOLUTION: The claim is true.

   PROOF: Suppose $n > 12$. Then $n/11 > 1$, so (by the definition of floor) $\lfloor n/11 \rfloor \geq 1$. Also, the definition of floor and ceiling mean that $\lfloor n/11 \rfloor \leq n/11 \leq \lceil n/11 \rceil$. Thus it only remains to prove that $\lceil n/11 \rceil < n$. Let $n = 11k + j$, where $j, k \in \mathbb{Z}$, and $0 \leq j < 11$ (j and k exist, by the division algorithm). Then

   $$\left\lceil \frac{n}{11} \right\rceil = \left\lceil \frac{11k + j}{11} \right\rceil = \begin{cases} k + 1, & j > 0 \\ k, & j = 0 \end{cases} \leq \frac{n + 10}{11}.$$

   Since $n > 12$, certainly $n > 10$ and $2n > 10 + n$. Thus $11n > n + 10$, so $n > (n + 10)/11 \geq \lceil n/11 \rceil$, as wanted. Putting the pieces together, $1 \leq \lfloor n/11 \rfloor \leq \lceil n/11 \rceil < n$. QED.

2. $\forall n \in \mathbb{N} - \{0\}, 2^{\lceil \log_2 n \rceil} < 2n$.

   SAMPLE SOLUTION: The claim is true.

   PROOF: $2n = 2 \times 2^{\log_2 n} = 2^{(\log_2 n)+1}$. Since log is strictly monotonic, it only remains to prove that $\lceil \log_2 n \rceil < (\log_2 n) + 1$. Suppose not, in other words, suppose $\lceil \log_2 n \rceil \geq (\log_2 n) + 1$. Then we'd have (subtracting 1 from both sides) $\lceil \log_2 n \rceil - 1 \geq \log_2 n$, contradicting the definition of $\lceil \log_2 n \rceil$ as the smallest integer that is no smaller than $\log_2 n$. Therefore $\lceil \log_2 \rceil < (\log_2 n) + 1$, and we have:

   $$2^{\lceil \log_2 n \rceil} < 2^{(\log_2 n)+1)} = 2n.$$