CSC236 QUIZ 4, TUESDAY JUNE 14TH

Name: Student number:

Recall the language of well-formed arithmetic expressions, $\mathcal{E}$ defined in class as the smallest set such that:

1. x, y, z are in $\mathcal{E}$ (this is the basis).
2. If e_1 and e_2 are in $\mathcal{E}$, then so are $(e_1 + e_2)$, $(e_1 - e_2)$, $(e_1 \times e_2)$, and $(e_1 \div e_2)$ (this is the induction step).

Use structural induction to prove that the number of left-parentheses ["("] equals the number of right-parentheses [")"] for every $e \in \mathcal{E}$.

SAMPLE SOLUTION: Let $lp(e)$ be the number of left parentheses in expression e and $rp(e)$ be the number of right parentheses in e.

CLAIM: For all $e \in \mathcal{E}$, $lp(e) = rp(e)$.

PROOF (STRUCTURAL INDUCTION ON e): If e is defined in the basis, then $e \in \{x, y, z\}$, so $lp(e) = 0 = rp(e)$, since e has no parentheses. Thus the claim holds for the basis.

INDUCTION STEP: Assume that the claim holds for e_1 and e_2, arbitrary elements of $\mathcal{E}$, and $e = (e_1 \oplus e_2)$, where $\oplus \in \{+, -, \times, \div\}$. Thus the number of left parentheses in e is the sum of those in e_1 and e_2, plus one extra left parenthesis that begins the expression. Similarly, the number of right parentheses in e is the sum of those in e_1 and e_2, plus one right parenthesis that ends the expression, so

$$\begin{aligned} lp(e) &= 1 + lp(e_1) + lp(e_2) \quad \text{[by observation above]} \\ &= 1 + rp(e_1) + rp(e_2) \quad \text{[by IH]} \\ &= rp(e) \quad \text{[by observation above]} \end{aligned}$$

Thus $P(\{e_1, e_2\}) \Rightarrow P(e)$, since $(e_1 \oplus e_2)$ is an arbitrary expression formed during the induction step, and thus the induction step preserves $P(e)$.

I conclude that $P(e)$ is true for all $e \in \mathcal{E}$. QED.