# CSC236, Summer 2005, Assignment 4

Due: Thursday July 28th, 10 am

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## Instructions

Please work on all questions. Turn in the outline and structure of a proof, even if you cannot provide every step of the proof, and we will try to assign some part marks. However, if there is any question you cannot see how to even begin, leave it blank and you will receive 20% of the marks for that question.

Be sure to give full credit to any sources you consult (other than course notes, TAs, and the instructor) in preparing this problem set. If you try to pass off somebody else's work as your own for credit, you are committing an academic offense, and that can entail serious consequences. Any ideas that you do not attribute to someone else are assumed to be the ideas of the author(s) listed below, and will be evaluated for grading.

Write your name(s) and student number(s) (maximum of two names and two student numbers) in the space below.

Name ______________________________

Student # ______________________________

Name ______________________________

Student # ______________________________

1. Let $x$, $y$, and $z$ be propositional variables and let $n$ be a natural number. Using only the connectives $\vee, \wedge$, and $\rightarrow$ and the given variables, how many different propositional formulae are there with exactly $n$ connectives?

   For $n = 0$ there are 3 formulae, namely $x$, $y$, and $z$.

   For $n = 1$ there are 27 formulae.

   For $n = 2$ there are 486 formulae...

   Derive a general formula for the number of such propositional formulae having $n$ connectives. Prove that your formula is correct.

2. In the propositional formulae below I use the rules of precedence from chapter 5 of the Course Notes to reduce the number of parentheses. In your solution you are welcome to re-introduce parentheses if it makes things clearer.

   (a) Use the logical equivalences in section 5.6 (no truth tables) to prove that
   $(P \rightarrow (Q \leftrightarrow R))$ LEQV $\neg(P \wedge ((Q \rightarrow R) \vee (R \rightarrow Q)) \wedge \neg(\neg Q \wedge \neg R) \wedge \neg(Q \wedge R))$

   (b) Write a CNF formula equivalent to both formulae in part (a). Prove that your CNF formula is correct using the logical equivalences from Section 5.6.

   (c) The *Sheffer's stroke* (or *nand*) operator, $|$, is a binary connective defined on page 138 of the Course Notes. Is $|$ associative? Prove your claim.

   (d) Let $P_1, P_2, \ldots P_n$ and $Q$ be arbitrary propositional formulae. Prove that for any $n \geq 2$,

   $$P_1 \vee P_2 \vee \ldots \vee P_n \rightarrow Q$$

   is logically equivalent to

   $$\neg Q \rightarrow \neg P_1 \wedge \neg P_2 \wedge \ldots \wedge \neg P_n$$

   by induction on $n$.

3. The Course Notes mention that the $|$ (*nand*) and $\downarrow$ (*nor*) operators form complete sets of connectives by themselves. We've also seen the binary operators $\rightarrow$, $\vee$, $\wedge$, $\leftrightarrow$ and $\oplus$ (exclusive or), each of which cannot form a complete set of connectives by itself. The 7 boolean functions represented by these connectives are frequently used in mathematics, logic, and computer science. There *are* other boolean operators, but they don't seem to get as much attention.

   (a) Two binary boolean operators (or functions) $\odot$ and $\diamond$ are *distinct* if there exist propositional formulae $P$ and $Q$ such that $P \odot Q$ is *not* logically equivalent to $P \diamond Q$. The 7 operators listed above are all distinct. How many distinct boolean operators are there in total? Explain your answer.

   (b) A binary boolean operator $\odot$ is *trivial* if for all propositional formulae $P$, $Q$, $R$, and $S$, $P \odot Q$ LEQV $R \odot S$. How many distinct trivial boolean operators are there? Explain your answer.

   (c) A binary boolean operator $\odot$ is *one-sided* if for all propositional formulae $P$, $Q$ and $S$, either $P \odot Q$LEQV$P \odot S$ or $P \odot Q$LEQV$S \odot Q$. How many distinct one-sided boolean operators are there? Explain your answer.

   (d) How many non-trivial, non-one-sided operators are missing from the above list of seven? Could we use the missing operators as a complete set of connectives? Prove your claims.

4. Let $\mathcal{LG}$ be a first-order language having an infinite set of variables including $a$, $b$, and $c$, predicate $P$ of arity 3, and predicate $\approx$ (the equality predicate). Consider an interpretation where the domain $D$ is the set of all vertices of some (simple) graph $G$, and $P(a, b, c)$ is true if and only if $b$ lies on a shortest simple path from $a$ to $c$ in $G$ [1].

   (a) Give a formula expressing the claim: "there are exactly three vertices in $G$". Explain what your formula says in precise English.

   (b) Give a formula expressing the claim "there exists an edge from vertex $a$ to vertex $b$ in $G$". Explain what your formula says in precise English.

   (c) Give a formula expressing the claim "$G$ is a tree"[2]. Explain what your formula says in precise English.

[1] Recall that a path from $a$ to $b$ is a sequence of vertices $v_1, v_2, \ldots v_k$ such that each two adjacent vertices are connected by an edge, $a = v_1$ and $b = v_k$. A *simple* path is a path such that all $k$ vertices in the path are distinct (not equal). A *shortest* simple path from $a$ to $b$ is a simple path such that no other path from $a$ to $b$ has fewer vertices (smaller $k$). Note that a path from $a$ to $b$ includes both $a$ and $b$, so $P(a, a, b)$ and $P(a, b, b)$ are always true if there is a path from $a$ to $b$. Also note that every path runs in both directions (on a simple graph) so $P(a, b, c) \rightarrow P(c, b, a)$.

[2] A tree is a graph which is connected (every vertex has at least one path to every other vertex) and acyclic (every vertex has no more than one path to any other vertex).