CSC236, Summer 2005, Assignment 2 sample solution

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1. MANIPULATE A STACK: Suppose you have a sequence of n distinct characters, and a LIFO (Last In, First Out) stack that allows exactly two operations:

   (a) PUSH: If the sequence is nonempty, remove the first element from the sequence and add it to the top of the stack. Otherwise do nothing.

   (b) POPP: If the stack is nonempty, remove the top element and print it to output. Otherwise do nothing.

   If you begin with a sequence of $n = 2$ distinct characters, then you can produce exactly 2 distinct outputs. Suppose your sequence is $\langle xy \rangle$, then you can produce

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   xy: push popp push popp
   yx: push push popp popp
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   How many different outputs can you produce with a sequence xyz, of length 3? How about of length n? Prove your claims.

   CLAIM: Let $f(n)$ be defined as

   $$f(n) = \begin{cases} 1, & n = 0 \\ \sum_{i=0}^{n-1} f(i)f(n-1-i), & n > 0 \end{cases}.$$

   CLAIM: Let $P(n)$ be "There are $f(n)$ distinct outputs from the stack described above starting with a string with n distinct characters." Then for all $n \in \mathbb{N}, P(n)$.

   PROOF (COMPLETE INDUCTION ON n): If $n = 0$, then $P(0)$ asserts that there is $f(0) = 1$ distinct output starting with an empty string. This is certainly true, since the unique empty string is output, so the base case holds.

   INDUCTION STEP: Assume that $P(\{0, \ldots, n-1\})$ is true for some arbitrary natural number n. I need to prove that this implies $P(n)$ is true. If $n = 0$ there is nothing to prove, since this was verified in the base case. Otherwise the IH assume $P(i)$ and $P(n-1-i)$ for every $0 \leq i \leq n-1$. WLOG, assume that the first character of the original sequence of length n is the character x, and partition the output sequences according to where x occurs in the output — at position i of the output, where $0 \leq i \leq n-1$. This partition counts all possible outputs, and has no duplicates, since a particular output is specified by the position of the character x.
   Since this is a LIFO stack, the i characters that are output before x, in positions $\{0, \ldots, i-1\}$, must have been pushed onto the stack after x was pushed, and popped from the stack before x was popped. Thus these characters are the next i characters pushed following x in the original sequence, that is characters $\{1, \ldots, i\}$ of the original sequence. Since they are pushed and

popped through a stack with x sitting on the bottom, by the IH they have $f(i)$ distinct outputs.

Similarly the $n-1-i$ characters that are output after x, in positions $\{i+1,\ldots,n-1\}$, are both pushed and popped after x is popped, which means they pushed and popped after the i characters output before x. Thus these characters follow the first i in the original sequence, so they are characters $\{i+1,\ldots,n-1\}$ of the original sequence. Since they are pushed and popped through a stack that starts out empty (after x is popped), by the IH they have $f(n-1-i)$ distinct outputs.

Let F_i be the set of distinct outputs of the first i characters following x in the original sequence, and F_{n-1-i} be the set of distinct outputs of characters $\{i+1,\ldots,n-1-i\}$ of the original sequence. The Cartesian product $F_i \times F_{n-1-i}$ has $f(i)f(n-1-i)$ pairs (see Chapter 0 of the Course Notes). There is a 1–1 correspondence between the pairs of outputs in $F_i \times F_{n-1-i}$ and the outputs of length n with x in position i, simply by concatenating the first element of the pair with x and then with the second element of the pair. Thus there are $f(n)f(n-1-i)$ distinct outputs of length n with x in position i.

Summing these over all the partitions, for each possible position x may occupy in the output, yields $\sum_{i=0}^{n-1} f(i)f(n-1-i)$ possible outputs. Thus $P(\{0,\ldots,n-1\}) \Rightarrow P(n)$, as wanted.

I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

2. Here is a recursive definition for $\mathcal{T}^*$, a subset of the family of ternary strings. Let $\mathcal{T}^*$ be the smallest set such that:

BASIS: 0 is in $\mathcal{T}^*$.

INDUCTION STEP: If $x, y \in \mathcal{T}^*$, then so are $x0y$, $1x2$, and $2x1$.

(a) Prove that if $k \in \mathbb{N}$, then there is no string in $\mathcal{T}^*$ with exactly $3^k + 1$ zeros.

CLAIM A1: Let $P(e)$ be "e has an odd number of zeros." Then for all $e \in \mathcal{T}^*, P(e)$.

PROOF (INDUCTION ON e): Suppose e is in the basis. Then $e = 0$, which has an odd number of zeros, so the claim holds for the basis.

INDUCTION STEP: Suppose x, y are arbitrary elements of $\mathcal{T}^*$. There are three cases to consider

i. $e = x0y$, then by the induction hypothesis for some $j, k \in \mathbb{Z}$, x has $2k+1$ zeros and y has $2j+1$ zeros. Thus e has $2(j+k+1)+1$ zeros, which is an odd number since $j+k+1$ is an integer (the integers are closed under addition).

ii. $e = 1x2$, then by the induction hypothesis x has an odd number of zeros, which is the same number as e does, since e adds no zeros.

iii. $e = 2x1$, then by the induction hypothesis x has an odd number of zeros, which is the same number as e does, since e adds no zeros.

Thus in all three possible cases, e has an odd number of zeros, so $P(\{x,y\}) \Rightarrow P(e)$.
I conclude that $P(e)$ is true for all $e \in \mathcal{T}^*$. QED.

CLAIM A2: Let $P(k)$ be "$3^k + 1$ is even." Then for all $k \in \mathbb{N}, P(k)$.

PROOF (INDUCTION ON k): If $k = 0$ then $P(k)$ states that $3^0 + 1 = 2$ is even, which is certainly true, so the claim holds for the base case.

INDUCTION STEP: Assume $P(k)$ for some arbitrary $k \in \mathbb{N}$. I must show that this implies $P(k+1)$. By the IH, $3^k + 1$ is even, so there is some integer i such that $3^k + 1 = 2i$. This means that $3^{k+1} + 1$ can be written as

$$\begin{aligned} 3^{k+1} + 1 &= 3(3^k) + 1 = 3(2i-1) + 1 \qquad \text{[by IH]} \\ &= 6i - 2 = 2(3i-1). \end{aligned}$$

Since $3i - 1$ is an integer (integers are closed under multiplication and subtraction), $3^{k+1} + 1$ is even, and so $P(k) \Rightarrow P(k+1)$.

I conclude that $P(k)$ is true for all $k \in \mathbb{N}$. QED.

By A2, if expression e has $3^k + 1$ zeros, then e has an even number of zeros, hence not an odd number of zeros. By A1, every expression in $\mathcal{T}^*$ has an odd number of zeros, so $e \notin \mathcal{T}^*$. QED.

(b) Prove that if $k \in \mathbb{N}$, then there is no string in $\mathcal{T}^*$ that has exactly 2^{k+1} digits.

CLAIM B1: Let $P(e)$ be "e has an odd number of digits." Then $\forall e \in \mathcal{T}^*, P(e)$.

PROOF (INDUCTION ON e): Suppose e is defined in the basis. Then $e = 0$, and hence has 1 digit, which is odd, so the claim holds for the basis.

INDUCTION STEP: Assume that $P(x)$ and $P(y)$ hold for arbitrary expressions in $\mathcal{T}^*$. There are three cases to consider:

i. If $e = x0y$, then the number of digits in e is the sum of the number of digits in x and y, plus one more digit. Thus, for some integers j, k expression e has $2j + 1 + 2k + 1 + 1$ digits, which can be rewritten as $2(j + k + 1) + 1$ digits. This is an odd number, since $(j + k + 1)$ is an integer (integers are closed under addition). Thus in this case $P(\{x, y\}) \Rightarrow P(e)$.

ii. If $e = 1x2$, then the number of digits in e is the sum of the number of digits in x plus 2. Thus, for some integer k, e has $2k + 1 + 2$ digits, or $2(k + 1) + 1$ digits, an odd number since $(k + 1)$ is an integer. Thus, in this case, $P(\{x, y\}) \Rightarrow P(e)$.

iii. If $e = 2x1$, then the number of digits in e is the sum of the number of digits in x plus 2. Thus, for some integer k, e has $2k + 1 + 2$ digits, or $2(k + 1) + 1$ digits, an odd number since $(k + 1)$ is an integer. Thus, in this case, $P(\{x, y\}) \Rightarrow P(e)$.

In all three cases, $P(\{x, y\}) \Rightarrow P(e)$, and these cases exhaust the possibilities, so $P(\{x, y\}) \Rightarrow P(e)$ for an arbitrary expression defined in the induction step.

I conclude that $P(e)$ is true for all $e \in \mathcal{T}^*$. QED.

Suppose some string e has 2^{k+1} digits, for some $k \in \mathbb{N}$. Then (re-writing) that e has 2×2^k digits, an even number (since 2^k is an integer). Thus e does not have an odd number of digits, so $P(e)$ is false, so by B1, $e \notin \mathcal{T}^*$. QED.

(c) Prove that there is no string in $\mathcal{T}^*$ whose digits sum to 97.

CLAIM C1: Let $P(e)$ be "The digits of e sum to an integer multiple of 3." Then $\forall e \in \mathcal{T}^*, P(e)$.

PROOF (STRUCTURAL INDUCTION ON e): If e is defined in the basis, then $e = 0$, and its digits sum to $0 = 3 \times 0$, which is an integer multiple of 3. Thus $P(e)$ holds for the basis.

INDUCTION STEP: Assume that $P(x)$ and $P(y)$ hold for arbitrary elements of $\mathcal{T}^*$. There are three cases to consider:

i. If $e = x0y$, then the sum of the digits in e is the sum of the digits in x plus 0 plus the sum of the digits in y. Thus, by the IH, for some integers j, k, the sum of the digits in e is $3j + 3k + 0 = 3(j + k)$, which is an integer multiple of 3, since $(j + k)$ is the sum of integers, and hence an integer. So in this case $P(\{x, y\}) \Rightarrow P(e)$.

ii. If $e = 1x2$, then the sum of the digits in e is 1 plus the sum of the digits in x plus 2. Thus, by the IH, for some integer k, the sum of the digits in e is $1 + 3k + 2 = 3(k + 1)$, which is a multiple of 3 since $(k + 1)$ is the sum of integers (and hence an integer). So in this case $P(\{x, y\}) \Rightarrow P(e)$.

iii. If $e = 2x1$, then the sum of the digits in e is 2 plus the sum of the digits in x plus 1. Thus, by the IH, for some integer k, the sum of the digits in e is $2 + 3k + 1 = 3(k + 1)$, which is a multiple of 3 since $(k + 1)$ is the sum of integers (and hence an integer). So in this case $P(\{x, y\}) \Rightarrow P(e)$.

The three cases are exhaustive, and in each case $P(\{x, y\}) \Rightarrow P(e)$, so $P(\{x, y\}) \Rightarrow P(e)$.

I conclude that $P(e)$ is true for all $e \in \mathcal{T}^*$. QED.

According to Proposition 1.7 of the Course Notes, any natural number has a unique quotient and remainder when divided by 3. In the case of 97 the quotient is 32 and the remainder is 1, whereas any multiple of 3 has a remainder of 0, so 97 is not a multiple of 3. Suppose a string e has 97 characters. Since 97 is not an integer multiple of 3, $P(e)$ is false, so by C1 $e \notin \mathcal{T}^*$. QED.

3. In lecture we discussed the recursive formula for $G(n)$, the number of binary strings of length n that do not have adjacent zeros.

   (a) Using the expression from class, derive a closed form for $G(n)$, the number of binary strings of length n that do not have adjacent zeros.

   SOLUTION: The formula we derived in class is:

$$G(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ G(n-1) + G(n-2), & n > 1 \end{cases}.$$

A short proof by induction would establish that this formula gives the number of binary strings of length n that do not have adjacent zeros, but you are allowed to assume the formula given. Comparing $G(n)$ to $F(n)$ (the Fibonacci function) shows that $G(0) = F(2)$ and $G(1) = F(3)$. We would like to prove that, in general, $G(n) = F(n + 2)$. Let $P(n)$ be "$G(n) = F(n + 3)$."

CLAIM: $\forall n \in \mathbb{N}, P(n)$.

PROOF (INDUCTION ON n): If $n = 0$, then $P(n)$ asserts that there are $F(2) = 1$ binary strings of length 0 without adjacent zeros, which is certainly true since the unique length-zero binary string doesn't have adjacent zeros. If $n = 1$, then $P(1)$ asserts that there are $F(3) = 2$ binary strings of length 1 without adjacent zeros, and this is certainly true since both binary strings of length 1 do not have adjacent zeros. Thus the claim holds for the basis.

INDUCTION STEP: Assume that $P(\{0, \ldots, n-1\})$ is true for some arbitrary natural number n. I want to show that this implies $P(n)$. If $n < 2$ there is nothing to prove, since we have shown that $P(n)$ holds in the base case. Otherwise, the induction hypothesis claims that $P(n-1)$ and $P(n-2)$ are both true, so

$$\begin{aligned} G(n) &= G(n-1) + G(n-2) \quad \text{[assumed defn. of } G(n) \text{ for } n > 1] \\ &= F(n+1) + F(n) \quad \text{[induction hypothesis]} \\ &= F(n+2) \quad \text{[definition of } F(n = 2)] \end{aligned}$$

Thus $P(\{0, \ldots, n-1\}) \Rightarrow P(n)$.

I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

We already have a closed form for $F(n)$, and we can now use it to express a closed form for $G(n)$:

$$G(n) = F(n+2) = \frac{\phi^{n+2} - \widehat{\phi}^{n+2}}{\sqrt{5}},$$

...where $\phi = (1 + \sqrt{5})/2$, and $\widehat{\phi} = (1 - \sqrt{5})/2$.

(b) Using the approach from class, develop a recursive formula (but not a closed form) for $H(n)$, the number of binary strings of length n that do not have 3 adjacent zeros. Justify your formula.

CLAIM: Define $H(n)$ by

$$H(n) = \begin{cases} 2^n, & n < 3 \\ H(n-1) + H(n-2) + H(n-3), & n > 2 \end{cases}.$$

Let $P(n)$ be "There are $H(n)$ binary strings of length n without 3 adjacent zeros." Then $\forall n \in \mathbb{N}, P(n)$.

PROOF (INDUCTION ON n): For $n \in \{0, 1, 2\}$ $P(n)$ asserts there are 2^n binary strings of length n without 3 adjacent zeros. This is certainly true since there are (established in the Course Notes) 2^n binary strings of length n, and if $n < 3$ all of these do not have 3 adjacent zeros. Thus $P(n)$ holds for the base case.

INDUCTION STEP: Assume that $P(\{0, \ldots, n-1\})$ holds for some arbitrary integer n. I want to prove that this implies $P(n)$. If $n < 3$, we're done, since $P(n)$ was established in the base case. Otherwise, the IH assume $P(n-1)$, $P(n-2)$ and $P(n-3)$. To count the number of binary strings without 3 adjacent zeros, we partition them into three disjoint sets:

i. The binary strings of length n without 3 adjacent zeros with final digit 1. These are formed by appending a 1 to any binary string of length $n-1$ that doesn't have 3 adjacent zeros, so there are $H(n-1)$ of these by the IH.

ii. The binary strings of length n without 3 adjacent zeros that end with the string 10. These are formed by appending 10 to any binary string of length $n-2$ that doesn't have 3 adjacent zeros, so there are $H(n-2)$ of these by the IH.

iii. The binary strings of length n without 3 adjacent zeros that end with the string 100. These are formed by appending 100 to any binary string of length $n-3$ that doesn't have 3 adjacent zeros, so there are $H(n-3)$ of these by the IH.

The three cases are exhaustive and disjoint, so there are $H(n-1) + H(n-2) + H(n-3)$ binary strings of length n without 3 adjacent zeros, so $P(\{0, \ldots, n-1\}) \Rightarrow P(n)$, as wanted.

I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED. Thus $H(n)$ is the number of binary strings of length n that don't have 3 adjacent zeros.

(c) Find a closed form for $J(n)$, which is defined for $n \in \mathbb{N}$ as:

$$J(n) = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ J(n-1) + 2J(n-2), & n > 1 \end{cases}.$$

SOLUTION: The first step is to seek a real number that obeys the given recurrence, that is find r so that $r^n = r^{n-1} + 2r^{n-2}$. Dividing by r^{n-2} yields the quadratic equation:

$$r^2 - r - 2 = 0.$$

This equation has roots $r_0 = 2$ and $r_1 = -1$, and any linear combination of these roots satisfies the recurrence, so for $n > 1$, $\alpha r_0^n + \beta r_1^n = \alpha r_1^{n-1} + \beta r_1^{n-1} + 2(\alpha r_0^{n-2} + \beta r_1^{n-2})$. We solve for α and β by considering the initial conditions, $J(0)$ and $J(1)$:

$$\begin{aligned} \alpha r_0^0 + \beta r_1^0 &= J(0) = 1 \Longrightarrow \beta = 1 - \alpha \\ \alpha r_0^1 + \beta r_1^1 &= 2\alpha - \beta = 2\alpha - (1 - \alpha) = 3\alpha - 1 = 1 \Longrightarrow \alpha = 2/3, \beta = 1/3. \end{aligned}$$

This yields a closed form for $J(n)$:

$$J(n) = \frac{2}{3}2^n + \frac{1}{3}(-1)^n = \frac{2^{n+1} + (-1)^n}{3}.$$

Let $P(n)$ be "$J(n) = (2^{n+1} + (-1)^n)/3$."

CLAIM: For all $n \in \mathbb{N}$, $P(n)$.

PROOF (INDUCTION ON n): If $n = 0$, then $P(0)$ asserts that $J(n) = 1 = (2^1 + (-1)^0)/3$, which is certainly true. If $n = 1$, then $P(1)$ asserts that $P(1) = 1 = (2^2 - 1)/3$, which is certainly true. So $P(n)$ holds for the base cases.

INDUCTION STEP: Assume that $P(\{0, \ldots, n-1\})$ hold for some arbitrary integer n. I want to show that this implies $P(n)$. If $n < 2$ there is nothing to prove, since $P(n)$ was verified in the base case. Otherwise the IH assumes that $P(n-1)$ and $P(n-2)$ are true, so

$$\begin{aligned}
J(n) &= J(n-1) + 2J(n-2) && \text{[definition of } J(n) \text{ when } n > 1] \\
&= \frac{2^n + (-1)n - 1 + 2(2^{n-1} + (-1)^{n-2})}{3} && \text{[IH for } P(n-1) \text{ and } P(n-2)] \\
&= \frac{2^{n+1} + (-1)^{n-2}(-1+2)}{3} && \text{[combine terms]} \\
&= \frac{2^{n+1} + (-1)^2(-1)^n}{3} = \frac{2^{n+1} + (-1)^{n-2}}{3} && \text{[multiply by 1]}
\end{aligned}$$

Thus $P(\{0, \ldots, n-1\}) \Rightarrow P(n)$, as wanted.

I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. Thus $J(n) = (2^{n+1} + (-1)^n)/3$ for all $n \in \mathbb{N}$. QED.

4. HACK SOME ALGEBRA:

(a) The binomial coefficient $\binom{n}{k}$ is defined for nonnegative integers $0 \leq k \leq n$ by:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and it represents the number of ways of choosing k elements from a set of n elements. Use the definition of $\binom{n}{k}$ to prove that if $0 < k < n$, then:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

PROOF: Suppose k is some arbitrary positive natural number less than n. Then $n - 1 \geq k > k - 1 \geq 0$, so both $\binom{n-1}{k}$ and $\binom{n-1}{k-1}$ are defined, and we can use the given definition:

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} && \text{[by given definition]} \\
&= \frac{(n-k)(n-1)! + k(n-1)!}{k!(n-k)!} && \text{[common denominators]} \\
&= \frac{n!}{k!(n-k)!} = \binom{n}{k} && \text{[by given definition]}
\end{aligned}$$

Since k is an arbitrary positive natural number less than n, this proves the claim. QED.

(b) Prove that if $1 \leq k \leq n$, then

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

PROOF: Let k be an arbitrary natural number no greater than n, so $n, k > k-1 \geq 0$, so $\binom{n-1}{k-1}$ is defined, and

$$\begin{aligned}
k\binom{n}{k} &= k\frac{n!}{k!(n-k)!} && \text{[by given definition]} \\
&= \frac{n(n-1)!}{(k-1)!(n-1-[k-1])!} && \text{[divide by non-zero } n \text{ and } k\text{]} \\
&= n\binom{n-1}{k-1} && \text{[by given definition]}
\end{aligned}$$

Since k is an arbitrary positive natural number no less than n, this proves the claim. QED.

(c) Suppose $x, y \in \mathbb{R}$. Use induction on n and part (a) to prove that:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

CLAIM: Let $P(n)$ be "$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$." Then $P(n)$ is true for all $n \in \mathbb{N}$.

PROOF (INDUCTION ON n): If $n = 0$ then $P(n)$ claims that $(x+y)^0 = 1 = \sum_{k=0}^{0} \binom{0}{0} x^0 y^0$, which is certainly true since $x^0 y^0$ is 1 for arbitrary real numbers x and y. Thus the base case holds.

INDUCTION STEP: Assume that $P(n)$ is true for an arbitrary natural number n. I must prove that this implies $P(n+1)$. I can re-group $(x+y)^{n+1}$ and use the IH so that

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} && \text{[by IH]} \\
&= \sum_{j=0}^{n} \binom{n}{j} x^{j+1} y^{n-j} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1} && \text{[use variable } j \text{ in first sum]} \\
&= \binom{n}{0} x^0 y^{n+1} + \sum_{j=0}^{n-1} \binom{n}{j} x^{j+1} y^{n-j} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + \binom{n}{n} x^{n+1} y^0 \\
\text{[}k = j+1\text{]} &= \binom{n+1}{0} y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n-k+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + \binom{n+1}{n+1} x^{n+1} \\
\text{[Part (a)]} &= \binom{n+1}{0} y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n-k+1} + \binom{n+1}{n+1} x^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}
\end{aligned}$$

Thus $P(n) \Rightarrow P(n+1)$, for an arbitrary natural number n.
I conclude that $P(n)$ is true for every $n \in \mathbb{N}$. QED.

(d) Prove that

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

PROOF: Use the fact that the zeroth term of the sum is zero, and part (b), so

$$\begin{aligned}
\sum_{k=0}^{n} k\binom{n}{k} &= \sum_{k=1}^{n} k\binom{n}{k} \\
\text{[part (b)]} &= \sum_{k=1}^{n} n\binom{n-1}{k-1} \\
[j = k-1\text{, and part (c)]} &= n\sum_{j=0}^{n-1}\binom{n-1}{j}1^j 1^{n-1-j} = n(1+1)^{n-1} \\
&= n2^{n-1}.
\end{aligned}$$

Thus the claim holds for an arbitrary natural number n. QED.

(e) Suppose n is a positive integer. Use the previous parts and some manipulation of the sum to prove that:

$$\sum_{k=0}^{n} k\binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k} = 1.$$

PROOF: Let n be an arbitrary positive integer. Using the fact that the zeroth term of the sum vanishes, and part (b)

$$\begin{aligned}
\sum_{k=0}^{n} k\binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k} &= \sum_{k=1}^{n} k\binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k} \\
\text{[part (b)]} &= n\sum_{k=1}^{n}\binom{n-1}{k-1}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k} \\
\text{[factor out } 1/n\text{]} &= \frac{n}{n}\sum_{k=1}^{n}\binom{n-1}{k-1}\left(\frac{1}{n}\right)^{k-1}\left(\frac{n-1}{n}\right)^{n-k} \\
[j = k-1] &= \sum_{j=0}^{n-1}\binom{n-1}{j}\left(\frac{1}{n}\right)^j\left(\frac{n-1}{n}\right)^{n-1-j} \\
\text{[part (c)]} &= \left(\frac{1}{n}+\frac{n-1}{n}\right)^{n-1} = 1.
\end{aligned}$$

Thus the claim holds for an arbitrary positive integer n. QED.