Sample solution
CSC236, Summer 2005, Assignment 1

Due: Thursday June 9th, 10 am

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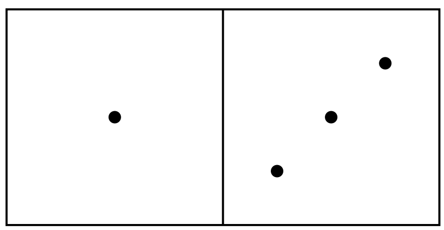
1. COUNT DOMINOES AND DROMINOES:

   (a) A set of dominoes has 28 tiles, one for each unordered pair of dot sets, where the dot sets run from size zero (or blank) to six dots. How many tiles would there be in an extended game of dominoes, where the sizes of the dot set sizes run from 0 to n? Prove your claim. (If you've never played dominoes, consult your instructor or TA). Here's a sketch of the domino for the unordered pair $\{1, 3\}$:



CLAIM: Let $P(n)$ be "If an extended game of dominoes has dot set sizes that run from 0 to n, then $(n+1)(n+2)/2$ tiles are required." Then $\forall n \in \mathbb{N}, P(n)$.

PROOF (INDUCTION ON n): Suppose $n = 0$. Then $P(n)$ claims there is $1(2)/2 = 1$ tile required, which is certainly true, since the only tile is the completely blank, i.e. $\{0, 0\}$ tile. So $P(0)$ is true.

INDUCTION STEP: Assume $P(n)$ is true for some arbitrary $n \in \mathbb{N}$, that is there are $(n+1)(n+2)/2$ tiles required for an extended game of dominoes where the dot set sizes run from 0 to n (this is the inductive hypothesis IH). I must show that this implies $P(n+1)$. Partition the tiles for a game of dominoes where the dot set sizes run from 0 to $n+1$ into those tiles that do not have a dot set of size $n+1$ and those that do. By the IH there are $(n+1)(n+2)/2$ tiles that do not have a dot set of size $n+1$, and a dot set of size $n+1$ must occur with every dot set size from 0 to $n+1$ exactly once — $n+2$ possibilities. This means there are:

$$\begin{aligned} \frac{(n+1)(n+2)}{2} + (n+2) &= \frac{(n+1)(n+2) + 2(n+2)}{2} \\ &= \frac{(n+2)(n+1+2)}{2} = \frac{(n+2)(n+3)}{2}. \end{aligned}$$

This is exactly $P(n+1)$, so $P(n) \Rightarrow P(n+1)$.

Thus, by induction, $P(n)$ is true for all $n \in \mathbb{N}$. QED.

   (b) Drominoes are like dominoes, except each tile is marked with an unordered triple instead of an unordered pair. How many dromino tiles are there if the triples run from $\{0, 0, 0\}$ to $\{6, 6, 6\}$? How about $\{0, 0, 0\}$ to $\{n, n, n\}$ (where n is an arbitrary natural number)? Prove your claims.

CLAIM: Let $P(n)$ be: "If dromino tiles run from $\{0,0,0\}$ to $\{n,n,n\}$, then there are $(n+1)(n+2)(n+3)/6$ dromino tiles required." Then, $\forall n \in \mathbb{N}, P(n)$.

PROOF (INDUCTION ON n): Suppose $n = 0$, then $P(n)$ claims there are $(1)(2)(3)/6 = 1$ dromino tile, which is correct since the only tile is the $\{0,0,0\}$ tile.

INDUCTION STEP: Assume $P(n)$ is true for some arbitrary $n \in \mathbb{N}$, thus there are $(n+1)(n+2)(n+3)/6$ drominoes if the tiles run from $\{0,0,0\}$ to $\{n,n,n\}$ (this is the IH). I must show that this implies $P(n+1)$, so I take a set of dromino tiles that run from $\{0,0,0\}$ to $\{n+1,n+1,n+1\}$, and I partition them into those tiles that have no $n+1$ (there are $(n+1)(n+2)(n+3)/6$ of these, by the IH), and those that have at least one $n+1$. In the latter partition I have exactly one tile that matches $n+1$ with each of the ordered pairs from $\{0,0\}$ to $\{n+1,n+1\}$, and by the previous result I know there are $(n+2)(n+3)/2$ of these. This means that in total there are

$$\begin{aligned}\frac{(n+1)(n+2)(n+3)}{6} + \frac{(n+2)(n+3)}{2} &= \frac{(n+1)(n+2)(n+3) + 3(n+2)(n+3)}{6} \\ &= \frac{(n+2)(n+3)(n+4)}{6},\end{aligned}$$

tiles, which is what $P(n+1)$ claims. Thus $P(n) \Rightarrow P(n+1)$.

Thus, by induction, $P(n)$ is true for all $n \in \mathbb{N}$. QED.

As a consequence of this claim, if a game of drominoes runs from $\{0,0,0\}$ to $\{6,6,6\}$, it requires $(7)(8)(9)/6 = 84$ tiles.

2. COUNT ACQUAINTANCES:

(a) On the last day of lecture of last Summer's CSC236 evening course, there were 51 students. Prove that there were at least two students with exactly the same number of student acquaintances attending the lecture.

CLAIM: There are at least two students with exactly the same number of student acquaintances attending the lecture.

PROOF: Let set A_i be the set of students with i student acquaintances. I need to show that there is some i in the range $0 \le i \le 50$ such that A_i has two or more students in it. There are two cases to consider.

CASE 1, THERE IS NO STUDENT IN A_0: If every student knows at least one other student, then the 51 students are distributed into 50 sets A_i, where $1 \le i \le 50$. By the pigeonhole principle (the students are pigeons and the sets are holes) there is at least one set with two or more students, so in this case the claim holds.

CASE 2, THERE IS AT LEAST ONE STUDENT IN A_0: Since there is at least one student who is acquainted with no other student, there is no student who is acquainted with every other student, so there is no student in A_{50}. Thus the 51 students are distributed into 50 sets A_i, where $0 \le i \le 49$, and by the pigeonhole principle there is at least one set with two or more students, so in this claim the claim holds as well.

The two cases cover all possibilities, and in both cases the claim holds. QED.

(b) During the break on the last day of lecture of last Summer's CSC236, more than six students lined up for beverages. Prove that among the first six students in the lineup there were either three mutual strangers or three mutual acquaintances (you are not allowed to assume that students are guaranteed to become acquaintances simply by standing in the same lineup). Is it also true that among the first five students in the line up there were either three mutual strangers or three mutual acquaintances? Prove your claim.

PROOF: Consider the first student in the lineup, s_1. There are two cases to consider:

s_1 HAS AT LEAST 3 ACQUAINTANCES AMONG THE NEXT 5 STUDENTS: Call s_1's 3 acquaintances s_2, s_3, and s_4. If s_2, s_3, s_4 are mutual strangers, then the claim is true, since we've found 3 mutual strangers. Otherwise at least 2 of them, WLOG s_2 and s_3, are acquaintances of each other and thus s_1, s_2, and s_3 are mutual acquaintances and the claim holds.

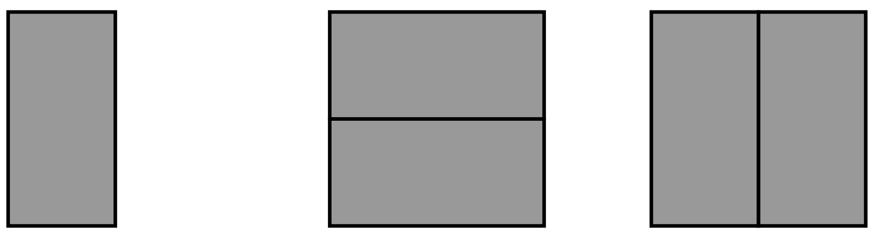
s_1 HAS FEWER THAN 3 ACQUAINTANCES AMONG THE NEXT 5 STUDENTS: In this case there are at least 3 students among the next 5 students who are strangers to s_1. Call the 3 students that s_1 doesn't know s_2, s_3, and s_4. If s_2, s_3, and s_4 are mutual acquaintances, then the claim is true, since we've found 3 mutual acquaintances. Otherwise, at least 2 of them, WLOG s_2 and s_3 don't know each other and s_1, s_2 and s_3 are mutual strangers and the claim holds.

The two cases cover all possibilities, and in both cases the claim holds. QED.

Consider the following possible configuration of students $\{s_1, s_2, s_3, s_4, s_5\}$, where the only students who are acquainted are: $\{s_1, s_2\}$, $\{s_1, s_5\}$, $\{s_2, s_3\}$, $\{s_3, s_4\}$, $\{s_4, s_5\}$. The other 5 2-sets of students are $\{s_1, s_3\}$, $\{s_1, s_4\}$, $\{s_2, s_4\}$, $\{s_2, s_5\}$, and $\{s_3, s_5\}$. In this configuration there are no triples of mutual acquaintances or mutual strangers, so the claim doesn't necessarily hold for the first 5 students.

3. Tiling with dominoes

Wooden dominoes are 2×1 rectangles, and if you turn them face down they have no identifying dots. In the diagram below I show (left) the one way to tile a 2×1 space with face-down dominoes, and (right) the two ways to tile a 2×2 space with dominoes.



Call the number of ways to tile a $2 \times n$ space with dominoes $D(n)$. Find a (preferably closed form) expression for $D(n)$. A closed form involves a fixed (not depending on n) number of well-understood operations. Prove that your form is correct.

CLAIM 1: Let $P(n)$ be "$D(n) = D(n-2) + D(n-1)$." Then, $\forall n \in \mathbb{N} \setminus \{0, 1, 2\}, P(n)$.

PROOF (COMPLETE INDUCTION ON n): Assume that $n > 2$ and $P(\{3, \ldots, n-1\})$ is true. This means that $D(n-2)$ and $D(n-1)$ are defined, and the inductive hypothesis states that we can tile a $2 \times n$ space in $D(n-1)$ ways by placing a vertical tile to the right of a tiling of a $2 \times (n-1)$ space, or in $D(n-2)$ ways by placing two horizontal tiles to the right of a tiling of a $2 \times (n-2)$ space. This gives a total of $D(n-1) + D(n-2)$ ways to tile a $2 \times n$ space, so $P(\{3, \ldots, n-1\}) \Rightarrow P(n)$.

Thus, by complete induction, $P(n)$ holds for all $n \in \mathbb{N} \setminus \{0, 2, 3\}$. QED.

But wait a minute! $D(n) = D(n-1) + D(n-2)$ is going to (recursively) create many operations $O(2^n)$ to "unwind," since you'll have to first find $D(n-1)$ and $D(n-2)$ the same way. That's not closed form. Time to check out the hint about Theorem 3.2.

CLAIM 2: For all $n \in \mathbb{N} \setminus \{0\}$, $D(n) = (\phi^{n+1} - \widehat{\phi}^{n+1})/\sqrt{5}$, where $\phi = (1 + \sqrt{5})/2$, and $\widehat{\phi} = (1 - \sqrt{5})/2$.

PROOF: By inspection of Equation 3.6 (page 78) it is clear that $D(1) = F(2)$ and $D(2) = F(3)$. This means that for $n > 0$, $D(n) = F(n+1)$, so by Theorem 3.2 $D(n) = F(n+1) = (\phi^{n+1} - \widehat{\phi}^{n+1})/\sqrt{5}$. QED.

Now that ugly mess IS in closed form, and curiously it is an integer for every $n > 0$.

4. DIVISION ALGORITHM:

(a) As part of Proposition 1.7 of the Course Notes there is a proof by induction that if $m, n \in \mathbb{N}$, and $n \neq 0$ then there are natural numbers q and r such that $m = qn + r$ and $r < n$. Use the Well-Ordering Principle for a different proof of this fact. HINT: Compare the equation $m = 0n + m$ to the equation $m = qn + r$. Notice that you aren't asked to prove the uniqueness of q and r.

PROOF: Let $m, n \in \mathbb{N}$, and $n \neq 0$. Let $R = \{t \in \mathbb{N} : \exists q \in \mathbb{N}, m = qn + t\}$. Since clearly $m = 0n + m$, $m \in R$, so R is non-empty. This means that by the Principle of Well-Ordering R contains a smallest element, which we'll call r, and there is some $q \in \mathbb{N}$ such that $m = qn+r$. It remains to show that $r < n$. Suppose not, then $r \geq n$, so $r = n + u$, for some $u \geq 0$. But then $m = qn + r = (q+1)n + u$, and $u \in R$, with $u = r - n < r$ (since $n \neq 0$), contradicting the fact that r is the smallest element of R. The contradiction shows the assumption that $r \geq n$ is false, so $r < n$. QED.

(b) Read the proof of Proposition 1.7, and notice that it is CONSTRUCTIVE — that is if you provide a natural number m and a positive natural number n, it tells you how to find a quotient q and a remainder r that satisfy $m = qn + r$, $0 \leq r < n$. Write a recursive java program that takes m and n as parameters and returns the pair (q, r). The structure of your program should mimic the proof of Proposition 1.7.

SAMPLE SOLUTION: Consider the following java class:

```
public class NumberFacts {

  public static int[] divMod(int m, int n) {
    int dm[] = new int[2];

    // This is the base case of Prop 1.7, m == 0
    // 0 == 0n + 0
    if (m == 0) {
      dm[0] = 0;
      dm[1] = 0;
    }

    // This is the induction step of Prop 1.7, m > 0,
    // so we assume divMod(m-1,n) returns the proper value
    else {
      // dm[0] = q', dm[1] = r', and m-1 == q'n + r', and n > r' >= 0
      dm = divMod(m-1,n);
      if (dm[1] < n-1) {
        // m = m-1+1 = q'n + (r' + 1), and (r' + 1) < n
         dm[1] += 1;}
      else { // dm[1] == n - 1 == q'
        // m == m-1+1 == q'n + r' + 1 == (q'+1)n + 0
        dm[0] += 1;
        dm[1] = 0;
      }
    }
    return dm;
  }

}
```

5. CONNECTIONS

In Section 0.5 you will find a definition of an undirected graph, in terms of its points and edges, as well as a definition of a PATH from point x to point y. A CONNECTED GRAPH is an undirected graph which has a path from x to y for every pair of points (x, y). Use complete induction to prove that a connected graph with n points has at least $n-1$ edges.

CLAIM: Let $P(n)$ be "A connected graph with n points has at least $n-1$ edges." Then $\forall n \in \mathbb{N}, P(n)$.

PROOF (INDUCTION ON n): Suppose $n = 0$. Then there are 0 edges, and $0 \geq -1 = 0 - 1$, so the claim holds. Suppose $n = 1$, then there are 0 edges, and $0 \geq 0 = 1 - 1$, so the claim holds. Thus the claim holds for $n \in \{0, 1\}$.

INDUCTION STEP: Suppose n is some arbitrary natural number greater than 1, and assume that $P(k)$ holds for every k, $0 \leq k < n$. We must use this to show that $P(n)$ holds. Let G be a connected graph with $n > 1$ nodes, and let x be some node of G. Since G is connected and has more than 1 node, there is at least one other node y, and a path from x to y. This means that x has at least one edge leading out from it, call these edges $\{(x, u_1), \ldots, (x, u_k)\}$, where $k \geq 1$. Remove edges $\{(x, u_1), \ldots, (x, u_k)\}$. Notice that this creates $j \leq k$ connected node-disjoint subgraphs (collections of vertices from G that remain connected) (since the subgraph connected to x via u_i remains connected to u_i), plus x itself.

Each of the connected subgraphs, G_i, has $n_i < n$ nodes, so the induction hypothesis applies and each connected subgraph has $e_i \geq n_i - 1$ edges. Other than x there are $n-1$ nodes, so we can get a lower bound on the number of edges, e, in G by using the IH on each connected component G_i:

$$\begin{aligned} e &\geq k + \sum_{i=1}^{j}(n_i - 1) \quad \text{[by IH]} \quad = n - 1 + k - j \\ &\geq n - 1 \qquad \text{[since } k \geq j\text{]}. \end{aligned}$$

This is exactly what $P(n)$ claims, so $P(\{0, \ldots, n-1\}) \Rightarrow P(n)$.

Thus, by complete induction, $P(n)$ is true for all $n \in \mathbb{N}$. QED.

6. BIG NUMBERS

For natural number n we define:

$$n! = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ n(n-1)!, & n > 1 \end{cases}$$

Use simple induction to prove that for all positive natural numbers n,

$$n! > \left(\frac{n}{e}\right)^n.$$

You may assume, without proof, that for any positive natural number n,

$$\left(1 + \frac{1}{n}\right)^n < e.$$

CLAIM: Let $P(n)$ be "$n! > (n/e)^n$." Then $\forall n \in \mathbb{N} \setminus \{0\}, P(n)$.

PROOF (INDUCTION ON n): Suppose $n = 1$. Then $n! = 1$, which is strictly greater than $(1/e)^1 = (n/e)^n$, so the claim holds for $n = 1$.

INDUCTION STEP: Assume that $P(n)$ is true for some arbitrary natural number greater than 0; that is $n! > (n/e)^n$. I must show that this implies $P(n+1)$. Since $n > 0$ I know that $n = (n+1)/(1+1/n)$, so

$$\begin{aligned}
(n+1)! = (n+1)n! &> (n+1)\left(\frac{n}{e}\right)^n \qquad \text{[by the IH]} \\
&= (n+1)\left(\frac{n+1}{(1+1/n)e}\right)^n \\
&= (n+1)\frac{(n+1)^n}{(1+1/n)^n e^n} \\
&> \frac{(n+1)^{n+1}}{e^{n+1}} \qquad \text{[since } (1+1/n)^n < e\text{]}.
\end{aligned}$$

This is exactly what $P(n+1)$ claims, so $P(n) \Rightarrow P(n+1)$.

Thus, by induction, $P(n)$ is true for every positive $n \in \mathbb{N}$. QED.