QUESTION 1. [5 marks]

Define \( f(n) \) as:

\[
f(n) = \sum_{i=0}^{2n+1} \tau^i.
\]

Prove that \( f(n) \) is divisible by 8 for all \( n \in \mathbb{N} \).

SOLUTION: This question closely resembles Assignment 1, Q6a. The only difference is the base here is 7, rather than 2.

CLAIM: \( P(n) \): “\( f(n) \) is divisible by 8” is true for all \( n \in \mathbb{N} \).

PROOF (SIMPLE INDUCTION ON \( n \)): \( P(0) \) asserts that \( \sum_{i=0}^{1} \tau^i \) is divisible by 8, or \( 7^0 + 7^1 = 8 \) is divisible by 8. This is certainly true, since \( 8 = 8 \times 1 + 0 \), so the base case holds.

INDUCTION STEP: I wish to show that for any \( n \), \( P(n) \implies P(n + 1) \), so I assume \( P(n) \) for an arbitrary \( n \in \mathbb{N} \), in other words I assume that \( \sum_{i=0}^{2n+1} \tau^i = 8k \), for some \( k \in \mathbb{N} \) (this is the IH).

Now I can break up the sum \( \sum_{i=0}^{2(n+1)+1} \tau^i \) and use the induction hypothesis:

\[
\sum_{i=0}^{2(n+1)+1} \tau^i = \tau^0 + \tau^1 + \left( \sum_{i=2}^{2n+1} \tau^i \right)
\]

[factor out \( \tau^2 \) ]

\[
= 8 + \left( \tau^2 \sum_{i=0}^{2n+1} \tau^i \right)
\]

[by IH] 

\[
= 8 + \left( \tau^2 8k \right) = 8(1 + \tau^2 k).
\]

Thus \( \sum_{i=0}^{2(n+1)+1} \tau^i \) is divisible by 8, so \( P(n) \implies P(n + 1) \), as wanted.

I conclude that \( P(n) \) holds for all \( n \in \mathbb{N} \). QED.

STATE AND VERIFY BASIS: 1 mark. -0.5 if \( f(n) \) is used as a predicate. -1 for making \( \forall n \) part of the predicate. -0.5 if the base case omits \( n = 0 \) (and starts at \( n = 1 \)).

SET UP INDUCTION: 1 mark. You need to state something equivalent to “I will show that \( P(n) \) implies \( P(n + 1) \),” or “assume \( P(n) \) for an arbitrary \( n \in \mathbb{N} \), now show \( P(n + 1) \).”

INDUCTION STEP: 2 marks. Show that \( P(n) \Rightarrow P(n + 1) \). -1 if step where IH is used is not explicitly shown.

CONCLUSION: 1 mark. Conclude that \( P(n) \) is true for all \( n \).
QUESTION 2.  [5 marks]

For \( n \in \mathbb{N} \), define \( B(n) \) as:

\[
B(n) = \begin{cases} 
1, & n = 0 \\
1, & n = 1 \\
B(n - 2) + B(n - 1), & n > 1 
\end{cases}
\]

Prove that \( B(n + 2) - \sum_{i=0}^{n} B(i) = 1 \) for all \( n \in \mathbb{N} \).

STATE AND VERIFY BASE CASE: 1 mark. -0.5 if you don’t state what claim your algebra is verifying.

SET UP INDUCTION: 1 mark. Either assume \( P(n) \) for some arbitrary \( n \), or say you will show that \( P(n) \Rightarrow P(n + 1) \). -1 mark for assuming \( P(n) \) for all \( n \).

INDUCTION STEP: Show that \( P(n) \Rightarrow P(n + 1) \). -1 mark if you don’t indicate where IH is used. -1 mark if you don’t indicate where definition of \( U \) is used.

CONCLUSION: Conclude that \( P(n) \) holds for all \( n \in \mathbb{N} \).

SOLUTION: This question closely resembles Assignment 2, Q2a. The difference is that the recursively-defined function has different starting conditions.

CLAIM: \( P(n) : "B(n) - \sum_{i=0}^{n} B(i) = 1" \) is true for all \( n \in \mathbb{N} \).

PROOF (SIMPLE INDUCTION ON \( n \)): \( P(0) \) asserts that \( B(2) - \sum_{i=0}^{0} B(i) = 1 \), or in other words \( 2 - 1 = 1 \), which is certainly true, so the base case holds.

INDUCTION STEP: In order to prove that for any \( n \in \mathbb{N} \), \( P(n) \Rightarrow P(n + 1) \), I assume \( P(n) \) for an arbitrary \( n \in \mathbb{N} \). In other words, my induction hypothesis (IH) is that \( B(n + 2) - \sum_{i=0}^{n} B(i) = 1 \).

Now I can re-write \( B(n + 1 + 2) - \sum_{i=0}^{n+1} B(i) \), and use the induction hypothesis

\[
B(n + 1 + 2) - \sum_{i=0}^{n+1} B(i) = B(n + 3) - \left( \sum_{i=0}^{n} B(i) \right) - B(n + 1)
\]

[by IH] \[
= B(n + 3) - (B(n + 2) - 1) - B(n + 1)
\]

[by definition of \( B(n + 3) \)] \[
= B(n + 3) - B(n + 3) + 1 = 1.
\]

Thus \( P(n) \) implies \( P(n + 1) \), as wanted.

I conclude that \( P(n) \) holds for all \( n \in \mathbb{N} \). QED.

STATE AND VERIFY BASE CASE: 1 mark. -0.5 if you don’t state what claim your algebra is verifying.

SET UP INDUCTION: 1 mark. Either assume \( P(n) \) for some arbitrary \( n \), or say you will show that \( P(n) \Rightarrow P(n + 1) \). -1 mark for assuming \( P(n) \) for all \( n \).

INDUCTION STEP: 2 marks. Show that \( P(n) \Rightarrow P(n + 1) \). -1 mark if you don’t indicate where IH is used.

-1 mark if you don’t indicate where definition of \( U \) is used.

CONCLUSION: 1 mark. Conclude that \( P(n) \) holds for all \( n \in \mathbb{N} \).
QUESTION 3.  [5 marks]

Let $PV = \{v, w, x, y, z\}$ be a set of propositional variables. Define a special set of propositional formulas $\mathcal{F}^*$ as the smallest set such that

**Basis:** Any propositional variable in $PV$ belongs to $\mathcal{F}^*$.

**Induction Step:** If $P_1$ and $P_2$ belong to $\mathcal{F}^*$, then so do $(P_1 \wedge P_2)$, $(P_1 \vee P_2)$, $(P_1 \rightarrow P_2)$ and $(P_1 \leftrightarrow P_2)$.

For a propositional formula $f$, define $cn(f)$ as the number of instances of connectives from $\{\lor, \land, \rightarrow, \leftrightarrow\}$ in $f$. Define $pv(f)$ as the number of instances of propositional variables from $\{v, w, x, y, z\}$ in $f$.

Use structural induction to prove that for all $f \in \mathcal{F}^*$, $pv(f) = cn(f) + 1$.

**Solution:** This question resembles the example worked in lecture (see lecture summary for Week 6).

**Claim:** $pv(f) = cn(f) + 1$ is true for all $f \in \mathcal{F}^*$.

**Proof (Structural induction on $f$):** For the basis, it is enough to check $f = u$, $f = v$, $f = x$, $f = y$, and $f = z$. In each case there is a single propositional variable and no connectives, so $pv(f) = 1 = cn(f) + 1$. Thus the base case holds.

**Induction Step:** Assume that $P(f_1)$ and $P(f_2)$ both holds, and that $f = (f_1 \ast f_2)$, where $\ast \in \{\land, \lor, \rightarrow, \leftrightarrow\}$. Notice that in each case, $f$ has the same number of propositional variables, and one more connective, than $f_1$ and $f_2$ do combined, so the following observations (observation 1 on the left, observation 2 on the right) hold:

$$ pv(f) = pv(f_1) + pv(f_2) \quad cn(f) = cn(f_1) + cn(f_2) + 1. $$

You can now combine these two observations to show:

- [by observation 1] $pv(f) = pv(f_1) + pv(f_2)$
- [by IH for $f_1$ and $f_2$] $cn(f_1) + 1 + cn(f_2) + 1$
- [by commutativity of addition] $cn(f_1) + cn(f_2) + 1 + 1$
- [by observation 2] $cn(f) + 1.$

This is exactly what $P(f)$ asserts, so $P(f_1)$ and $P(f_2)$ imply $P(f)$, as wanted.

I conclude that $P(f)$ holds for all $f \in \mathcal{F}^*$. QED.

**State and Verify Basis:** 0.5 marks. Check that $P(f)$ holds when $f$ is a propositional variable.

**Set Up Induction:** 1 mark. Show the connection between a new formula and formulas about which the property, $P$, is assumed.

**Induction Step:** 3 marks. Show that $P(f_1)$ and $P(f_2)$ imply $P(f)$. -1 for not indicating where IH is used. -0.5 if observations about number of connectives, parentheses, or variables in $f$ versus those in subformulas are not explained.

**Conclusion:** 0.5 marks. Conclude that property $P$ holds for all $f \in \mathcal{F}^*$. 

Remarks: There were some attempts to use simple induction on $pv(f)$ (this won’t work). There were some incorrect basis cases (not propositional variables).

Total Marks = 15