PROOF STRUCTURES
(WITH SOME COMMENTS ABOUT FILLING THEM IN)

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To Prove/Conclude

<u>$A \to B$</u>

Suppose A.
  ⋮
  Then B.
Thus $A \to B$.

If A is not just a predicate, expand it with the rules in When Assumed/Known (the section below). Put the expansion immediately after the "Suppose A".

If B is not just a predicate, expand it with the rules in this section; put the expansion immediately before the "Then B".

In general, all the rules require this: decide what they require you to prove and what they let you assume, then proceed recursively.

If we were *filling in the proof* (not just giving the structure), we would have the "indirect" *option* of proving the contrapositive instead.

Suppose $\neg B$.
  ⋮
  Then $\neg A$.
Then $\neg B \to \neg A$.
Thus $A \to \neg B$.

<u>$A \wedge B$</u>

⋮
Then A.
⋮
Then B.
Thus $A \wedge B$.

<u>$A \leftrightarrow B$</u>

We start with its definition:

⋮
Then $(A \to B) \wedge (B \to A)$.
Thus $A \leftrightarrow B$.

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We can begin the recursive expansion: the $\vdots$ indicate a proof of $(A \to B) \wedge (B \to A)$, which is of the form $() \wedge ()$, so

> $\vdots$
> Then $A \to B$.
> $\vdots$
> Then $B \to A$.
> Thus $A \leftrightarrow B$.

This contains two implications to prove, so

> Suppose A
>> $\vdots$
>> Then B.
>
> Then $A \to B$.
> Suppose B.
>> $\vdots$
>> Then A.
>
> Then $B \to A$.
> Thus $A \leftrightarrow B$.

<u>$\forall x \in D, B$</u>

> Let $x \in D$.
>> $\vdots$
>> Then B.
>
> Thus, since $x \in D$ is arbitrary and B: $\forall x \in D, B$.

The form $\forall x \in D, A \to B$ is very common, and when you expand it using the rule for $\forall$ and then the rule for $\to$, you may put the "Suppose A" at the same indentation level as "Let $x \in D$" if you like.

<u>$\exists x \in D, B$</u>

> Let $x =$___.
>> $\vdots$
>> Then $x \in D$.
>> $\vdots$
>> Then B.
>
> Thus, since $x \in D$ and B: $\exists x \in D, B$.

<u>$A \vee B$</u>

> Case: _C_
>> $\vdots$
>> Then A.
>
> Case: _$\neg C$_
>> $\vdots$
>> Then B.
>
> In each case, A or B.
> Thus, since (at least) one of the cases is true:$A \vee B$.

If we were filling in the proof we would need to choose C. We would have the option of more than one case C_1, C_2, ..., C_n as long as it is clear (or we also prove) that $C_1 \vee C_2 \vee \cdots \vee C_n$. We can also conclude just $A \vee B$ in one or both of the cases, but typically the point of choosing C is that it is a case where we can determine A specifically.

<u>$\neg B$</u>

You can either 'push' the negation inside, using our various DeMorgans laws or the meaning of B if it is a predicate, or use contradiction:

> Suppose, for contradiction, B.
>
>   ⋮
>
>   Then ___, a contradiction.
>
> Thus $\neg B$.

When Assumed/Known

<u>$A \wedge B$</u>

> $A \wedge B$.
> Then A.
> Then B.

<u>$\exists x \in D, B$</u>

> $\exists x \in D, B$.
> Let $x \in D$ such that B.

Question: How could you write the second line in two lines, using another term instead of "such that".

<u>$A \vee B \to R$</u>

$A \vee B$ alone does not automatically expand. But as the hypothesis in an implication:

> $A \vee B$.
> <u>Case: A</u>
>   ⋮
>   Then R.
> <u>Case: B</u>
>   ⋮
>   Then R.
> In each case, R.
> Since $A \vee B$, (at least) one of the cases is true, thus R.

<u>$A \to B$</u>

> $A \to B$.

This one does not automatically expand.

If we were filling in the proof, after $A \to B$ if we conclude A we may conclude B:

> $A \to B$.
> ⋮
> Then A.
> Thus also B.

When filling in the proof we also have the option of using the contrapositive:

> $A \rightarrow B$.
> Then $\neg B \rightarrow \neg A$
> $\vdots$
> Then $\neg B$.
> Thus also $\neg A$.

<u>$A \leftrightarrow B$</u>

> $A \leftrightarrow B$.
> Then $(A \rightarrow B) \wedge (B \rightarrow A)$.

Now expand the $\wedge$ appropriately.

<u>$\forall x \in D, B(x)$</u>

Traditionally people don't expand this (remember, we're not proving it). But here's a way to think about it (you may do this in your structures if you like).

> $\forall x \in D, B(x)$.
> Then $B(d_1) \wedge B(d_2) \wedge B(d_3) \ldots$,where the d_is are the elements of D.

How to use this is discussed in detail in the lecture notes about "Reusing Results".

<u>$\neg B$</u>

This doesn't expand well.

Instead, we have the option of 'pushing' the $\neg$ inside B with DeMorgan's Laws.