QUESTION 1. [5 MARKS]

Consider the following sentences, where d is in the domain D of dogs:

$N(d)$ means "d is nervous."

$B(d)$ means "d barks."

$T(d)$ means "d is a terrier." (Note: a terrier is a kind of dog).

Assume $S1 : \forall d \in D, T(d) \Rightarrow (N(d) \Rightarrow B(d))$. Are the following statements true or false, based on your assumption? Briefly justify your answers.

1. Every dog that doesn't bark is not nervous.

   SAMPLE SOLUTION: Doesn't follow from S1. It is false if you assume there is at least one nervous non-terrier that doesn't bark, true otherwise. -1 mark if you claimed either T or F without specifying your assumptions.

2. Every dog that isn't nervous is either not a terrier or doesn't bark.

   SAMPLE SOLUTION: Doesn't follow from the S1. It is false if you assume there is at least one calm terrier that barks, true otherwise. -1 mark if you claimed either T or F without specifying your assumptions.

3. Every dog that doesn't bark is either not nervous or not a terrier.

   SAMPLE SOLUTION: True. $S1$ is equivalent to $\forall d \in D, (T(d) \wedge N(d)) \Rightarrow B(d)$, so the contrapositive is $\forall d \in D, \neg B(d) \Rightarrow (\neg T(d) \vee \neg N(d))$. -1 mark if you said F, or T without justification.

4. Every nervous dog that doesn't bark is not a terrier.

   SAMPLE SOLUTION: True. Suppose d is nervous and doesn't bark, in other words $N(d)$ and $\neg B(d)$, so $N(d) \Rightarrow B(d)$ is false, since the antecedent is true while the consequent is false. This implication is the consequent of $S1$, so (since $S1$ is assumed to be true) the antecedent of $S1$, namely $T(d)$, must be false. -1 mark if you said F, or T without justification.

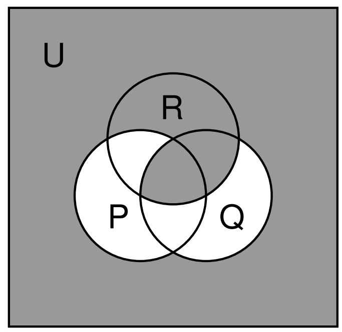
5. Every terrier both barks and is nervous.

   SAMPLE SOLUTION: Doesn't follow from S1. It is false if you assume there is at least one calm terrier, true otherwise. -1 mark if you said T or F without specifying your assumptions.

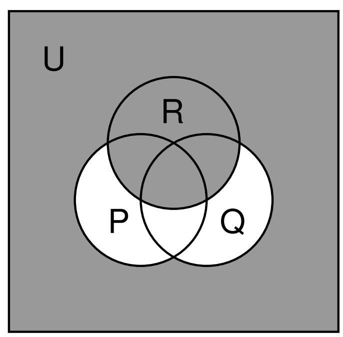
QUESTION 2. [5 MARKS]

Let U be some universe containing sets P, Q, R, and let $P(u)$ mean $u \in P$, $Q(u)$ means $u \in Q$, and $R(u)$ mean $u \in R$. For each statement in precise symbolic notation, shade the corresponding Venn diagram to indicate which regions can be non-empty without making the statement false. You earn more marks the more regions you shade correctly, and marks will be deducted if you shade regions that make the corresponding statement false. No justification required.

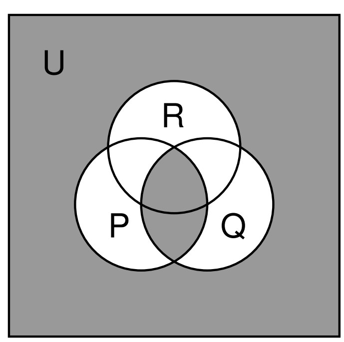
1. $\forall u \in U, (P(u) \Rightarrow Q(u)) \wedge (Q(U) \Rightarrow R(u))$.



2. $\forall u \in U, (P(u) \Rightarrow R(u)) \wedge (Q(u) \Rightarrow R(u))$.



3. $\forall u \in U, (P(u) \Leftrightarrow Q(u)) \wedge (R(u) \Rightarrow P(u))$



MARKING SCHEME: $-5/12$ mark for each region shaded when it should be blank, or blank when it should be shaded.

QUESTION 3. [5 MARKS]

Let $\mathbb{N}$ be the natural numbers $\{0, 1, 2, \ldots\}$, and $\mathbb{Z}$ be the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Use the direct proof format from lecture to prove that $\forall n \in \mathbb{N}$, n^2 odd $\Rightarrow n$ odd. You may assume, and use, the following claims to build your proof, as well as any other standard results about $\mathbb{Z}$, and $\mathbb{N}$ you wish to cite:

C1: $\forall m \in \mathbb{Z}$, m is odd $\Leftrightarrow$ m is not even.

C2: $\forall m \in \mathbb{Z}$, m is odd $\Leftrightarrow$ $\exists k \in \mathbb{Z}$ such that $m = 2k + 1$.

C3: $\forall m \in \mathbb{Z}$, m is even $\Leftrightarrow$ $\exists j \in \mathbb{Z}$ such that $m = 2j$.

C4: Products and sums of integers are integers also.

HINT: You may find it easier to prove the contrapositive of the given claim.

SAMPLE SOLUTION: Claim: $\forall n \in \mathbb{N}$, n^2 odd $\Rightarrow n$ odd.

PROOF: I use the format for direct proof, but I apply it to the contrapositive: $\forall n \in \mathbb{N}$, n not odd $\Rightarrow n^2$ not odd.

Let $n \in \mathbb{N}$

Assume n is not odd

Then n is even (claim C1)

So $\exists k \in \mathbb{Z}$ such that $n = 2k$ (claim C3)

So $n^2 = 4k^2 = 2(2k^2)$ (definition of n^2 and associativity)

Let $j = 2k^2$. Then $j \in \mathbb{Z}$ (claim C4) and $n^2 = 2j$

So n^2 is even (claim C3)

Thus n^2 is not odd (claim C1)

Hence n not odd $\Rightarrow n^2$ not odd.

So n^2 odd $\Rightarrow n$ odd (contrapositive)

Since n is an arbitrary natural number, $\forall n \in \mathbb{N}$, n^2 odd $\Rightarrow n$ odd.

MARKING SCHEME: -1 mark if your proof is not in the structured format from lecture. -2 marks if you claimed n^2 odd $\Rightarrow n^2 = 4k^2 + 4k + 1$ without justification. -1 mark if you didn't justify $2j^2 \in \mathbb{Z}$. -4 marks if you proved the converse. -1 mark if you omitted "hence (thus) n even $\Rightarrow n^2$ even" concluding the section where n is assumed even. -1 mark for any bi-implication step that is not justified.

Total Marks = 15