CSC165H, Mathematical expression and reasoning for computer science
week 9

15th August 2005

Gary Baumgartner and Danny Heap
heap@cs.toronto.edu
SF4306A
416-978-5899
http://www.cs.toronto.edu/~heap/165/S2005/index.shtml

How large is "sufficiently large?"

Is binary search a better algorithm than linear search?[1] It depends on the size of the input. For example, suppose you established that linear search has complexity $L(n) = 3n$ and binary search has complexity $B(n) = 9n \log_2 n$. For the first few n, $L(n)$ is smaller than $B(n)$. However, certainly for $n > 7$, $B(n)$ is smaller, indicating less "work" for binary search.

When we say "large enough" n, we mean we are discussing the asymptotic behaviour of the complexity function, and we are prepared to ignore the behaviour near the origin.

Making Big-O precise

Here's precise definition of "The set of functions that, ignoring a constant, are eventually no more than f."

For any function $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, let

$$O(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geq 0} | \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)\}.$$

Saying $g \in O(f)$ says that "g grows no faster than f" (or equivalently, "f is an upper bound for g), so long as we modify our understanding of "growing no faster" and being an "upper bound" with the practice of ignoring constant factors. Now we can prove some theorems.

Suppose $g(n) = 3n^2 + 2$ and $f(n) = n^2$. Then $g \in O(f)$. We need to prove that $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2 + 2 \leq cn^2$. It's enough to find some c and B that "work" in order to prove the theorem.

Finding c means finding a factor that will scale n^2 up to the size of $3n^2 + 2$. $c = 3$ almost works, but there's that annoying additional term 2. Certainly $3n^2 + 2 < 4n^2$ so long as $n \geq 2$, since $n \geq 2 \Rightarrow n^2 > 2$. So pick $c = 4$ and $B = 2$ (other values also work, but we like the ones we thought of first). Now concoct a proof of

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2 + 2 \leq cn^2.$$

Let $c = 4$.
Then $c \in \mathbb{R}^+$.

> Let $B = 2$.
>
> Then $B \in \mathbb{N}$.
>
> > Let $n \in \mathbb{N}$.
> >
> > Suppose $n \geq B$.
> >
> > > Then $n^2 \geq B^2 = 4$. (squaring is monotonic on natural numbers.)
> > > So $n^2 > 2$.
> > > So $3n^2 + n^2 > 3n^2 + 2$. (adding $3n^2$ to both sides of the inequality).
> > > So $3n^2 + 2 \leq 4n^2$.
> >
> > Thus, $n \geq B \to 3n^2 + 2 \leq 4n^2$.
> > Since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \geq B \to 3n^2 + 2 \leq 4n^2$.
>
> Since B is a natural number, $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \to 3n^2 + 2 \leq cn^2$.

Since c is a positive real number, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \to 3n^2 + 2 \leq cn^2$.

So, by definition, $g \in O(f)$. Now suppose that $g(n) = n^4$ and $f(n) = 3n^2$. Is $g \in O(f)$? No. We can see intuitively that any constant that we multiply times $3n^2$ will be overwhelmed by the extra factor of n^2 in $g(n)$. But to show this clearly, we negate the definition and then prove the negation:

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n^4 > c3n^2.$$

The parameter we have some control over is n, and we need to pick it so that $n \geq B$ and $n^4 > c3n^2$. Solve for n:

$$\begin{aligned} n^4 &> c3n^2 \\ \Rightarrow n^4/n^2 &> c3n^2/n^2 \\ \Rightarrow n^2 &> 3c \\ \Rightarrow n &> \sqrt{3c}. \end{aligned}$$

To satisfy the conditions, set $n = B + \lceil\sqrt{3c}\rceil + 1$. Since $\sqrt{3c}$ is not necessarily a natural number, we take its ceiling. Now we can generate the proof.

Let $c \in \mathbb{R}^+$. Let $B \in \mathbb{N}$.

Let $n = B + \lceil\sqrt{3c}\rceil + 1$.

Then $n \in \mathbb{N}$. (since $B \in \mathbb{N}$, $1 \in \mathbb{N}$, and $\lceil\sqrt{3c}\rceil \in \mathbb{N}$ (since $c > 0$) and $\mathbb{N}$ is closed under sums).
So $n \geq B$ (since it is the sum of B and two other non-negative numbers).
So $n \geq \sqrt{3c} + 1$. (since $B \geq 0$)
So $n^2 > (\lceil\sqrt{3c}\rceil + 1)^2$.
So $n^2 > 3c$. (ignoring some positive terms).
So $n^4 > 3cn^2$.

Since n is a natural number, $\exists n \in \mathbb{N}, n \geq B \wedge n^4 > c3n^2$.

Since c is an arbitrary element of $\mathbb{R}^+$ and B is an arbitrary element of $\mathbb{N}$, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n^4 > c3n^2$.

By definition, this means that $g \notin O(f)$.

Other bounds

In analogy with $O(f)$, consider two other definitions:

$$\Omega(f) \quad = \quad \{f : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0} | \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow g(n) \geq cf(n)\}.$$

To say $g \in \Omega(f)$ expresses the concept that g grows at least as fast as f." (f is a lower bound on g).

$$\Theta(f) \quad = \quad \{g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0} | \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n)\}.$$

To say "$g \in \Theta(f)$" expresses the concept that "g grows at the same rate as f." (f is a tight bound for g).

Some theorems

Here are some general results that we now have the tools to prove.

- $f \in O(f)$.
- $(f \in O(g) \wedge g \in O(h)) \Rightarrow f \in O(h)$.
- $g \in \Omega(f) \Leftrightarrow f \in O(g)$.
- $g \in \Theta(f) \Leftrightarrow g \in O(f) \wedge g \in \Omega(f)$.

Test your intuition about Big-O by doing the "scratch work" to answer the following questions:

- Are there functions f, g such that $f \in O(g)$ and $g \in O(f)$ but $f \neq g$?[2]
- Are there functions such that $f \notin O(f)$, and $g \notin O(g)$?[3]

To show that $(f \in O(g) \wedge g \in O(h)) \Rightarrow f \in O(h)$, we need to find a constant $c \in \mathbb{R}^+$ and a constant $B \in \mathbb{N}$, that satisfy:

$$\forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq ch(n).$$

Since we have constants that scale h to g and then g to f, it seems clear that we need their product to scale g to f. And if we take the maximum of the two starting points, we can't go wrong. Making this precise.

Assume $f \in O(g) \wedge g \in O(h)$.

So $f \in O(g)$.

So $g \in O(h)$.

So $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n > B \Rightarrow f(n) \leq cg(n)$. (by defn. of $f \in O(g)$).

Let $c_g \in \mathbb{R}^+, B_g \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c_g g(n)$.

So $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq ch(n)$. (by defn. of $g \in O(h)$).

Let $c_h \in \mathbb{R}^+, B_h \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_h \Rightarrow g(n) \leq c_h h(n)$.

Let $c = c_g c_h$. Let $B = \max(B_g, B_h)$.

Let $n \in \mathbb{N}$.

Suppose $n \geq B$.

Then $n \geq B_h$ (definition of max), so $g(n) \leq c_h h(n)$.
Then $n \geq B_g$ (definition of max), so $f(n) \leq c_g g(n) \leq c_g c_h h(n)$.
So $f(n) \leq ch(n)$.

So $n \geq B \Rightarrow f(n) \leq ch(n)$.

Since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq ch(n)$.

Since c is a positive real number, since B is a natural number, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq ch(n)$.

So $f \in O(g)$, by definition.

So $(f \in O(g) \wedge g \in O(h)) \Rightarrow f \in O(h)$

Notes

[1]Better in the sense of time complexity.

[2]Sure, $f = n^2$, $g = 3n^2 + 2$.

[3]Sure. f and g don't need to both be monotonic, so let $f(n) = n^2$ and

$$g(n) = \begin{cases} n, & n \text{ even} \\ n^3, & n \text{ odd} \end{cases}$$

So not every pair of functions from $\mathbb{N} \to \mathbb{R}^{\geq 0}$ can be compared using Big-O.