CSC165H, Mathematical expression and reasoning for computer science
week 4

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Those superscripts

In these course summaries you'll often find superscripts.[1] These often indicate answers to questions worked out in lecture, and through the wonders of word processing, those answers are formatted as endnotes (at the end of the document). My motivation isn't so much to give you whiplash moving your gaze between the question and the answer, as to allow you to form your own answer before looking at my version.

Multiple quantifiers

Many sentences we want to reason about have a mixture of predicates. For example

Claim 1: Some female employee makes more than 25,000.

We can make a few definitions, so let E be the set of employees, $\mathbf{Z}$ be the integers, $\mathbf{sm}(e,k)$ be e makes a salary of more than k, and $\mathbf{f}(e)$ be e is female. Now I could rewrite:

Claim 1': $\exists e \in E, \mathbf{f}(x) \wedge \mathbf{sm}(e, 25000)$.

It seems a bit inflexible to combine e making a salary, and an inequality comparing that salary to 25000, particularly since we already have a vocabulary of predicates for comparing numbers. We can refine the above expression so that we let $\mathrm{s}(e,k)$ be e makes salary k. Now I can rewrite again:

Claim 1": $\exists e \in E, \exists k \in \mathbf{Z},\ \mathbf{f}(x) \wedge \mathrm{s}(e,k) \wedge k > 25000$.

Notice that the following are all equivalent to Claim 1"[2]:
So $\wedge$ is commutative and associative, and the two existential quantifiers commute.

AND

We use $\wedge$ ("and") to combine two sentences into a new sentence that claims that both of the original sentences are true. In our employee database:

CLAIM 2: The employee makes less than 75,000 and more than 25,000.

Claim 2 is true for Al (who makes 60,000), but false for Betty (who makes 500). If we identify the sentences with predicates that test whether objects are members of sets, then the new $\wedge$ predicate tests whether somebody is in both the set of employees who makes less than 75,000 and the set of employees who make more than 25,000 — in other words, in the intersection. Is it a coincidence that $\wedge$ resembles $\cap$(only more pointy)?

We need to be careful with everyday language where the conjunction "and" is used not only to join sentences, but also to "smear" a subject over a compound predicate. In the following sentence the subject "There" is smeared over "pen" and "telephone:"

CLAIM 3: There is a pen and a telephone.

If we let O be the set of objects, $p(x)$ mean x is a pen, and $t(x)$ mean x is a telephone, then the obvious meaning of Claim 3 is[3] (There is a pen and there is a telephone). But a pedant who has been observing the trend where phones become increasingly smaller and difficult to use might think Claim 3 means:[4] (There is a pen-phone).

Here's another example whose ambiguity is all the more striking since it appears in a context (mathematics) where one would expect ambiguity to be sharply restricted.

> The solutions are:
>
> $x < 10$ and $x > 20$
>
> $x > 10$ and $x < 20$

In the first case the author means the union of two sets in the first case, and the intersection in the second. We use $\wedge$ in the second case, and disjunction $\vee$ ("or") in the first case.

OR

The disjunction "or" (written symbolically as $\vee$) joins two sentence into one that claims that at least one of the sentences is true. For example

> The employee is female or makes less than 45,000.

This sentence is true for Flo (she makes 20,000 and is female), for Carlos (who makes less than 45,000), but false for Al (he's neither female, nor does he make less than 45,000). If we viewed this "or'ed" sentence as a predicate testing whether somebody belonged to at least one of "the set of employees who are female" or "the set of employees who earn less than 45,000," then it corresponds to the union. As a mnemonic, the symbols $\vee$ and $\cup$ resemble each other.

We use $\vee$ to include the case where more than one of the properties is true, that is we use an inclusive or. In everyday English we sometimes say "and/or" to specify the same thing that this course uses "or" for, since the meaning of "or" can vary in English. The sentence "Either we play the game my way, or I'm taking my ball and going home now," doesn't include both possibilities.

DeMorgan's Laws

These laws can be verified either by a truth table, or by representing the sentences as Venn diagrams and taking the complement.

Sentence $s_1 \wedge s_2$ is false exactly when at least one of s_1 or s_2 is false. Symbolically:[5]

Sentence $s_1 \vee s_2$ is false exactly when both s_1 and s_2 are false. Symbolically:[6] By using the associativity of $\wedge$ and $\vee$, you can extend this to conjunctions and disjunctions of more than two sentences.

Logical arithmetic

If we identify $\wedge$ and $\vee$ with set intersection and union, it is clear that they are associative and commutative, so

$$P \wedge Q \Leftrightarrow Q \wedge P \text{ and } P \vee Q \Leftrightarrow Q \vee P$$
$$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R \text{ and } P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$$

Maybe a bit more surprising is that we have distributive laws for each operation over the other:

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

We can also simplify expressions using IDENTITY and IDEMPOTENCY laws:

IDENTITY: $P \wedge (Q \vee \neg Q) \Leftrightarrow P \Leftrightarrow P \vee (Q \wedge \neg Q)$.

IDEMPOTENCY: $P \wedge P \Leftrightarrow P \Leftrightarrow P \vee P$

Mixed quantifiers

If you mix the order of existential and universal quantifiers, you may change the meaning of a sentence. Consider the table below that shows who respects who

	A	B	C	D	E	F
A	◇					
B		◇	◇	◇	◇	◇
C		◇	◇	◇	◇	◇
D		◇	◇	◇	◇	
E		◇	◇	◇		
F		◇	◇			

If we want to discuss this table symbolically, we can denote the domain of people by P, and the predicate "x respects y" by $r(x, y)$. Consider the following open sentence:

Claim 4: $\exists x \in P,\ r(x, y)$, that is "y is respected by somebody."

If we pre-pended the universal quantifier $\forall y \in P$ to Claim 4, would it be true? As usual, check each element of the domain, column-wise, to see that it is[7] Symbolically

Claim 5: $\forall y \in P, \exists x \in P, r(x, y)$,

or "Everybody has somebody who respects him/her." You can have different x's depending on the y, so although every column has a diamond in some row, it need not be the same row for each column. What would the predicate be that claims that some row works for each column, that a row is full of diamonds?[8] Now we have to check whether there is someone who respects everyone:

CLAIM 6: $\exists x \in P, \forall y \in P,\ r(x,y)$

You will find no such row. The only difference between Claim 5 and Claim 6 is the order of the quantifiers. The convention we follow is to read quantifiers from left to right. The existential quantifier involves making a choice, and the choice may vary according to the quantifiers we have already parsed. As we move right, we have the opportunity to tailor our choice with an existential quantifier (but we aren't obliged to).

IMPLICATION, BI-IMPLICATION, WITH $\neg, \vee$, AND $\wedge$

If we shade a Venn diagram so that the largest possible portion of it is shaded without contradicting the implication $P \Rightarrow Q$, we gain some insight into how to express implication in terms of negation and union. The region that we can choose object x from so that $P(x) \Rightarrow Q(x)$ is $\neg P \cup Q$ (if we interpret $\neg P$ as the complement of P), and this easily translates to $\neg P \vee Q$. This gives us an equivalence:

$$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

Now use DeMorgan's law to negate the implication:

$$\neg(P \Rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow (\neg\neg P \wedge \neg Q) \Leftrightarrow (P \wedge \neg Q)$$

You can use a Venn diagram or some of the laws introduced earlier to show that bi-implication can be written with $\wedge \vee$, and $\neg$:

$$(P \Leftrightarrow Q) \Leftrightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

DeMorgan's law tells us how to negate this:

$$\neg(P \Leftrightarrow Q) \Leftrightarrow \neg((P \wedge Q) \vee (\neg P \wedge \neg Q)) \Leftrightarrow \cdots \Leftrightarrow ((\neg P \wedge Q) \vee (P \wedge \neg Q))$$

MOVING NEGATION IN

Sometimes things become clearer when negation applies directly to the simplest predicates we are discussing. Consider

CLAIM 6: $\forall x, \exists y, P(x,y)$

What does it mean for Claim 6 to be false, i.e. $\neg(\forall x, \exists y, P(x,y))$? It means there is some x for which the remainder of the sentence is false:

$$\neg(\forall x, \exists y, P(x,y) \Leftrightarrow \exists x, \neg\exists y, P(x,y)$$

So now what does the negated sub-sentence mean? It means there are no y's for which the remainder of the sentence is true:

$$\exists x, \neg\exists y, P(x,y) \Leftrightarrow \exists x, \forall y, \neg P(x,y)$$

There is some x that for every y makes $P(x,y)$ false. As negation ($\neg$) move from left to right, it flips universal quantification to existential quantification, and vice versa. Try it on the symmetrical counterpart $\exists x, \forall y, P(x,y)$, and consider

$$\neg(\exists x, \forall y, P(x,y)) \Leftrightarrow \forall x, \neg\forall y, P(x,y)$$

If it's not true that there exists an x such that the remainder of the sentence is true, then for all x the remainder of the sentence is false. Considering the remaining subsentence, if it's not true that for all y the remainder of the subsentence is true, then there is some y for which it is false:

$$\neg(\exists x, \forall y, P(x,y)) \Leftrightarrow \forall x, \exists y, \neg P(x,y)$$

For every x there is some y that makes $P(x,y)$ false.

Try combining this with implication, using the rule we discussed earlier, plus DeMorgan's law:

$$\neg(\exists x, \forall y(P(x,y) \Rightarrow Q(x,y))) \Leftrightarrow \neg(\exists x, \forall y(\neg P(x,y) \vee Q(x,y)))$$

Notes

[1] Like this.

[2]

$$\begin{aligned} \exists k \in \mathbf{Z},\ \exists e \in E, \mathbf{f}(e) \wedge \mathbf{s}(e,k) \wedge k > 25000 \\ \exists e \in E,\ \mathbf{f}(e) \wedge (\exists k \in \mathbf{Z}, \mathbf{s}(e,k) \wedge k > 25000) \\ \exists e \in E,\ \mathbf{f}(e) \wedge (\exists k \in \mathbf{Z},\ \mathbf{s}(e,k) \wedge k > 25000) \end{aligned}$$

[3] $\exists x \in O,\ p(x) \wedge \exists x \in O,\ t(x)$, or even $\exists x \in O$, $\exists y \in O$, $p(x) \wedge q(x)$.

[4] $\exists x \in O,\ p(x) \wedge t(x)$

[5] $\neg(s_1 \wedge s_2) \Leftrightarrow (\neg s_1 \vee \neg s_2)$

[6] $\neg(s_1 \vee s_2) \Leftrightarrow (\neg s_1 \wedge \neg s_2)$

[7] True, theres a diamond in every column.

[8] If we were thinking of the row corresponding to x, then $\forall y \in P$, $r(x, y)$.