CSC165H, Mathematical expression and reasoning for computer science
week 3

2nd June 2005

Gary Baumgartner and Danny Heap
heap@cs.toronto.edu
SF4306A
416-978-5899
http://www.cs.toronto.edu/~heap/165/S2005/index.shtml

Universal quanitfication and implication again

So far we have considered an implication to be universal quantification in disguise:

Claim 1: If an employee is male, then he makes less than 55,000.

The English indefinite article "an" signals that this means "Every male employee makes less than 55,000," and this closed sentence is either true or false, depending on the domain of employees. Since there is quantification going on, it's natural to wonder what open sentence is being quantified.

Claim 2: If the employee is male, then he makes less than 55,000.

The English definite article "the" often signals and unspecified value, and hence an open sentence. We could transform Claim 2 into Claim 1 by prefixing it with "For every employee, ..."

Claim 2': For every employee, if the employee is male, then he makes less than 55,000.

This distinction is probably clearer in symbolic notation. Let E mean the set of employees, predicate $L(e)$ mean that employee e makes less than 55,000, and $M(e)$ mean that employee e is male. The Claims 1 and 2' correspond to[1], whereas Claim 2 (no prime) corresponds to[2]. Since the claim is about male employees, we are tempted to say $\forall m \in M, L(m)$, however we usually take the approach of setting our domain to the largest universe in which the predicates make sense. We don't want to avoid reasoning about non-males. How do you feel about verifying Claim 2 for all six values in E, which are true/false?[3]

Do you feel uncomfortable saying that the implications with false antecedents are true? Implications are strange, especially when we consider them to involve causality (which we don't in logic). Consider:

Claim 3: If it rains in Toronto on June 2 2006, then there are no clouds.

Is Claim 3 true or false? Would your answer change if you could wait a year? What if you waited a year and June 2 was a completely dry day in Toronto, is Claim 3 true or false?[4]

Vacuous truth

We use the fact that the empty set is a subset of any set. Let $x \in \mathbf{R}$ (the domain is the real numbers). Is the following implication true or false?

Claim 4: If $x^2 - 2x + 2 = 0$, then $x > x + 5$.

A natural tendency is to process $x > x+5$ and think "that's impossible, so the implication is false." However, there is no real number x such that $x^2 - 2x + 2 = 0$, so the antecedent is false for every real x. Whenever the antecedent is false and the consequent is either true or false, the implication as a whole is true. Another way of thinking of this is that the set where the antecedent is true is empty (vacuous), and hence a subset of every set. Such an implication is sometimes called vacuously true.

In general, if there are no Ps, we consider $P \Rightarrow Q$ to be true, regardless of whether there are any Qs. Another way of thinking of this is that the empty set contains no counterexamples. Use this sort of thinking to evaluate the claims:[5]

Claim 4: All employees making over 80,000 are female.

Claim 5: All employees making over 80,000 are male.

Claim 6: All employees making over 80,000 have supernatural powers.

Equivalence

Suppose Al quits the domain E. Consider the claim

Claim 6a: Every male employee makes between 25,000 and 45,000.

Is Claim 6a true? What is its converse?[6] Is the converse true? Draw a Venn diagram. The two properties describe the same set of employees; they are equivalent. In everyday language, we might say "An employee is male if and only if the employee makes between 25,000 and 45,000." This can be decomposed into two statements:

> An employee is male if the employee makes between 25,000 and 45,000.
>
> An employee is male only if the employee makes between 25,000 and 45,000.

Here are some other everyday ways of expressing equivalence:

- P iff Q ("iff" being an abbreviation for "if and only if").
- P is necessary and sufficient for Q.
- $P \Rightarrow Q$, and conversely.

You may also hear

- P [exactly / precisely] when Q

For example, if our domain is $\mathbf{R}$, you might say "$x^2 + 4x + 4 = 0$ precisely when $x = -2$." Equivalence is getting at the "sameness" (so far as our domain goes) of P and Q. We may define properties P and Q differently, but the same members of the domain have these properties (they define the same sets). Symbolically we write $P \Leftrightarrow Q$. So now

> An employee is male $\Leftrightarrow$ he makes between 25,000 and 45,000.

Consider the following

> Only female employees make less than 1,000.

This is the (true) converse of the (false):

> If an employee is female, then she makes less than 1,000.

The first statement can be re-written as its own contrapositive, as a statement about male employees:

> If an employee is male, then he does not make less than 1,000.

Summing up

- $P \Rightarrow Q$ tells us about Ps. It's converse tells us about non-Ps.
- The equivalence $P \Leftrightarrow Q$ can be decomposed into $P \Rightarrow Q \wedge Q \Rightarrow P$, so it tells us about Ps, Qs, non-Ps, and non-Qs.

In everyday langauge you will hear people confuse (sometimes deliberately) an implication with its converse:

> If you are a criminal, then you have something to hide.
>
> Suppose you have something to hide.
>
> Then you are a criminal.

Existential quantification

Consider another sort of quantification, existential quantification:

Claim 7: There is an employee who makes less than 15,000.

Claim 8: An employee makes more than 100,000.

Although the indefinite article "an" is used here, we don't take it to signal universal quantification in Claim 7, due to the phrase "There is." Claim 8 is a bit ambiguous, and would be clearer if re-written as "Some employee..." How do we prove Claim 7 true?[7] Express this in terms of sets.[8] How do you prove or disprove Claim 8?[9] What does this mean in terms of sets?[10]

Existential quantification can turn an open sentence into a closed sentence (statement): "For some employee, the employee makes less than 15,000." In symbols we write $\exists$, which we pronounce "there exists." If E means the set of employees, $L(e)$ means that e makes less than 15,000, then we can write (with increasing symbolic content):

- $\exists$ employee, the employee makes less than 15,000.
- $\exists$ employee e, e makes less than 15,000.
- $\exists e \in E, L(e)$.

In everyday language existential quantification is expressed as:

> There [is / exists] [a / an / some / at least one] ... [such that / for which] ..., or [For] [a / an / some / at least one] ...

Note that the English word "some" is always used INCLUSIVELY here, so "some object is a P" is true if every object is a P. When is $\exists x, P(x)$ false?[11] The truth values of $\neg\exists x, P(x)$ and $\forall x, \neg P(x)$ are the same. Apply negation again.[12] Saying Claim 8 is false is the same as saying "Every employee does not make more than 100,000." Recall our test of when $\forall x, P(x)$ is false.[13] When there's a counterexample. Existential quantifiers can restrict the domain being considered:

CLAIM 9: Some female employee makes more than 25,000.

CLAIM 10: There exists a male employee making less than 10,000.

For universal quantification we express the restriction with implication (similar to subset inclusion), for existential quantification we express the restriction with AND (symbolically $\wedge$), which is like intersection of sets. In general

- "Every P is also a Q" becomes $\forall x, P(x) \Rightarrow Q(x)$.
- "Some P is also a Q" becomes $\exists x, P(x) \wedge Q(x)$.

Now Claim 9 becomes "$\exists$ employee e, e is female and e makes less than 25,000." The existence of example Flo makes Claim 9 true. Claim 10 is false, because no male employee makes less than 10,000. In symbols $\forall$employees e, e male $\Rightarrow \neg(e$ makes less than 10000). The following are equivalent, and you should become comfortable with reasoning why they are:

- $\neg(\exists x, P(x) \wedge Q(x) \Leftrightarrow \forall x, (P(x) \Rightarrow \neg Q(x))$. In words "No P is a Q" is equivalent to "Every P is a non-Q."
- $\neg(\forall x, P(x) \Rightarrow Q(x)) \Leftrightarrow \exists x(P(x) \wedge \neg Q(x))$. In words "Not every P is a Q" is equivalent to "There is some P that is a non-Q."

CONJUNCTION, DISJUNCTION

Notice that, symbolically $P \wedge Q$ is true exactly when both P and Q are true, and false if only one of them is true and the other is false. Similarly $P \vee Q$ is true if either one (or both) of P or Q is true. More on this next time.

NOTES

[1] $\forall e \in E, M(e) \Rightarrow L(e)$

[2] $M(e) \Rightarrow L(e)$

[3]

- If Al is male, then Al makes less than 55,000.
- If Betty is male, then Betty makes less than 55,000.
- If Carlos is male, then Carlos makes less than 55,000.
- If Doug is male, then Doug makes less than 55,000.
- If Ellen is male, then Ellen makes less than 55,000.
- If Flo is male, then Flo makes less than 55,000.

[4] True, regardless of the cloud situation. In logic $P \Rightarrow Q$ is false exactly when P is true and Q is false. All other configurations of truth values for P and Q are true (assuming that we can evaluate whether P and Q are true or false).

[5] All these claims are true, although possibly misleading. Any claim about elements of the empty set is true, since there are no counterexamples.

[6] Every employee making between 25,000 and 45,000 is male.

[7] Look up the table entry for Betty.

[8] The set of employees making less than 15,000 is not empty.

[9] You have to check every employee. The absence of EXAMPLES (rather than COUNTEREXAMPLES) makes it false.

[10] The set of employees earning over 100,000 is empty.

[11] When $\forall x, \neg P(x)$ is true.

[12] So $\exists x, P(x)$ is equivalent to $\neg\forall x, \neg P(x)$. We could live without existential quantifiers, but would our life be as good?

[13] When $\exists x, \neg P(x)$ is true. Notice the symmetry with the test of when $\forall x, P(x)$ is false, that is when $\exists x, \neg P(x)$.