CSC165H, Mathematical expression and reasoning for computer science
week 2

26th May 2005

Gary Baumgartner and Danny Heap
heap@cs.toronto.edu
SF4306A
416-978-5899
http://www.cs.toronto.edu/~heap/165/S2005/index.shtml

Sentences, statements, and predicates

Recall the table of employees with their genders and salaries from last lecture:

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

Now consider the following claims:

1. The employee makes less than 55,000.
2. Every employee makes less than 55,000.

Can you decide whether both claims are true or false? What is the basic difference between the two types of claim?[1]

We can express this symbolically by letting $L(x)$ denote "x makes less than 55,000." $L(x)$ is called a PREDICATE (you may think of a predicate as a boolean function), and x is a variable representing an element of the domain. If E is the set of employees, Claim 1 is equivalent to "$L(x)$", and it neither true nor false since x is unspecified. Claim 2 is equivalent to "for all employees x, $L(x)$." The phrase "for all employees x" quantifies the variable x.

Claim 1 is called a SENTENCE. It may refer to unquantified objects (for example x). Once the objects are specified (substitutions are made for the variable(s)), the sentence is either true or false (but not both). Claim 2 is called a STATEMENT. It doesn't refer to any unquantified variables, and it is either true or false (not both). Every statement is a sentence, but not every sentence is a statement. If you want to make it explicit that a sentence refers to unquantified objects, you may call it an "open sentence." Thus a sentence is a statement if and only if it is not open. Universal quantification transformed Claim 1 into Claim 2, from an open sentence into a statement.

Symbolic notation

We can indicate universal quantification symbolically as $\forall$, read as "for all." This only makes sense if we specify the universe (domain) from which we are considering "all" objects. With this notation, Claim 2 can be written

> $\forall$ employees, the employee makes less than 55,000.

Things become clearer if we introduce a name for the unspecified employee:

> $\forall$ employees x, x makes less than 55,000.

Since this statement may eventually be embedded in some larger and more complicated structure, we can add to the brevity and clarity by adding a bit more notation. Let E denote the set of employees, and $L(x)$ denote the predicate "x makes less than 55,000." Now Claim 2 becomes:

> $\forall x \in E, L(x)$.

Dissecting implication

What does "every P is a Q" tell us? In our database example:

> Claim 3: If an employee is female, then she makes less than 55,000.

Claim 3 discusses three sets, E, the set of employees, F, the set of female employees, and L, the set of employees making less than 55,000. Claim 3 implicitly invokes universal quantification, so it is more than a claim about a particular employee. The Venn diagram FImpliesL (see the web page) indicates the situation corresponding to our table, by shading non-empty regions. If you had no access to either the table or the Venn diagram, but only knew the Claim 3 was true, what would you know about

1. F, the set of female employees? What do you know about Ellen, if you only know that Ellen is female?
2. L, the set of employess earning less than 55,000? What do you know about Betty (if you only know she's in L) or Carlos (if you only know he's in L)?
3. $\overline{F}$, the set of male employees? Think about both Doug and Al.
4. $\overline{L}$ (the complement of L), the set of employees making 55,000 or more.

Knowing "P implies Q" tells us nothing more about,[2] however it does tell us more about.[3]

Suppose you have a new employee Grnflx (from a domain short of vowels), plus our Venn diagram without any shading (see web page). Which region of the Venn diagram would you add Grnflx to in order to make Claim 3 false?[4] Once you've shaded that region, does it matter whether any of the other regions are shaded or unshaded?[5]

More symbols

We can write implication symbolically as $\Rightarrow$, read "implies." Now "P implies Q" becomes $P \Rightarrow Q$. Claim 3 could now be re-written as

> an employee is female $\Rightarrow$ that employee makes less than 55,000.

Contrapositive

The CONTRAPOSITIVE of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$ ($\neg$ is the symbol for negation). In English the contrapositive of "all P is/are Q" is "all non-Q is/are non-P." Put another way, the contrapositive of "P implies Q" is "non-Q implies non-P." The contrapositive of Claim 3 is

an employee doesn't make less than 55,000 $\Rightarrow$ that employee is not female.

or, given the structure of the domain E of employees:

an employee makes at least 55,000 $\Rightarrow$ that employee is male.

Does the contrapositive of Claim 3 tell us everything that Claim 3 itself does? Check the Venn diagram (web page again). Does every Venn diagram that doesn't contradict Claim 3 also not contradict the contrapositive of Claim 3?[6]

Can you apply the contrapositive twice? To do this it helps to know that applying negation ($\neg$) twice toggles the truth value twice (I'm not not going means I'm going). Thus the contrapositive of the contrapositive of $P \Rightarrow Q$ is the contrapositive of $\neg Q \Rightarrow \neg P$, which is $\neg\neg P \Rightarrow \neg\neg Q$, equivalent to $P \Rightarrow Q$.

Converse

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. In words, the converse of "P implies Q" is "Q implies P." An implication and its converse don't mean the same thing. Consider the Venn diagram FIMPLIESL. Would it work as a Venn diagram for LIMPLIESF?[7]

Consider an example where the (implicit) domain is the set of pairs of numbers, perhaps $\mathbf{R} \times \mathbf{R}$.

$$x = 1 \Rightarrow xy = y$$

- If we know $x = 1$, then we know $xy = y$.
- If we know $x \neq 1$, then we don't know anything about xy.
- If we know $xy = y$, then we don't know anything about x.
- If we know $xy \neq y$, then we know $x \neq 1$.

The contrapositive of Claim 4 is:

$$xy \neq y \Rightarrow x \neq 1.$$

Check the four points we knew from Claim 4, and see whether we know the same ones from the contrapositive (it may be helpful to read them in reverse order). What about the the converse?

$$xy = y \Rightarrow x = 1$$

with equivalent contrapositive

$$x \neq 1 \Rightarrow xy \neq y.$$

The converse of Claim 3 is not equivalent to Claim 3, for example consider the pair $(5, 0)$, that is $x = 5$ and $y = 0$. Indeed, Claim 3 is true, while its converse is false.

Implication in everyday English

Here are some ways of saying "P implies Q" in everyday language. In each case, try to think about what is being quantified, and what predicates (or perhaps sets) correspond to P and Q.

- If P, [then] Q.
  "If nominated, I will not stand."
  "If you think I'm lying, then you're a liar!"
- When[ever] P, [then] Q.
  "Whenever I hear that song, I think about ice cream."
  "I get heartburn whenever I eat supper too late."
- P is sufficient/enough for Q
  "Differentiability is sufficient for continuity."
  "Matching fingerprints and a motive are enough for guilt."
- Can't have P without Q
  "There are no rights without responsibilities."
  "You can't stay enrolled in CSC165 without a pulse."
- P requires Q
  "Successful programming requires skill."
- For P to be true, Q must be true / needs to be true / is necessary
  "To pass CSC165, A student needs to get 40 on the final."
- P only if / only when / Q
  "I'll go only if you insist."

For the antecedent (P) look for if,when, enough, sufficient. For the consequent (Q) look for then, requires, must, need, necessary, only if, when. In all cases, check whether the expected meaning in English matches $P \Rightarrow Q$.

Notes

[1]Claim 1 depends on who you mean by "The employee." If you specify Al, Claim 1 is false, but if you specify Ellen, Claim 1 is true. Claim 2 is quantified, so it depends on the entire universe of employees. Claim 2 is false because you can find at least 1 counterexample.

[2]$\overline{P}$ (the complement of P), and Q.

[3]P (we know it's a subset of Q), or any P is a Q, and $\overline{Q}$ (the complement of Q), we know it's a subset of $\overline{P}$. Any not-Q is a not-P.

[4]Add Grnflx to $F - L$ (F outside L). Now Grnflx is a counter-example to the claim that every female employee makes less than 55,000.

[5]No. Counter-example Grnflx makes the implication false, and adding other data doesn't change this.

[6]Yes. The only Venn diagram that contradicts Claim 3 or its contrapositive is one that has at least one element in F outside of L.

[7]No, because there is shading in $L - F$, indicating that region is non-empty and contains counter-examples (like Doug and Carlos).