CSC165H, Mathematical expression and reasoning for computer science week 13

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Truncation

We have looked at some errors caused by the necessity of computing on inexact inputs (either due to finite precision of measurement, or finite precision of representation). However, even if our inputs were completely exact, most algorithms do not compute exact values and then round them.

In practice, to add two numbers on a computer, the number with the smallest exponent is rounded to "align"[1] the decimal point with the number with the larger exponent, and then addition is carried out, for example:

$$9.8765 \times 10^3 + 7.6543 \times 10^0 \approx (9876.5 + 7.7) \times 10^1 = 9.8842 \times 10^3$$

The true value is never computed. All arithmetic is implemented as a compromise between using small amounts of time and storage, and losing as little information as possible. There are libraries that allow arbitrary amounts of precision, but in the end this compromise is still necessary.

For other functions, such as $\sqrt{1+x}$ or e^x, we borrow tools from calculus, for example the Taylor series.

$$\begin{aligned}
\sqrt{1+x} &= 1 + x/2 - x^2/8 + x^3/16 - 5x^4/128 + \cdots \\
e^x &= 1 + x^2/2! + x^3/3! + x^4/4! + \cdots
\end{aligned}$$

Eventually computation has to stop, so these series are TRUNCATED. Take care with the order of operations to avoid accumulated error and cancellation. In all cases, we're stuck with an approximation of an infinite series. TRUNCATION is defined as using an approximate formula (for example, a finite sub-series of a Taylor series) to compute a value. Calculus will provide you with a bound on how bad the approximation can be.

The truncation error is independent of the rounding error. Rounding error affects the input (use rounded x' instead of true x for the input) and the representation of the result. Truncation affects the computations performed (instead of computing true function $f(x')$, compute truncated function $\widehat{f}(x')$). Both sources contribute to give your total error.

ERROR SUMMARY:

WE WANT	WE GET
exact value: x	approximate value: x'
exact function: f	approximate function: $\widehat{f}$
exact result: $f(x)$	computed result: $\widehat{f}(x')$.

The absolute error for the entire computation has two sources of error. The first term is contributed by truncation or instability, the second term is due to rounding (in measurement or representation):

$$|\widehat{f}(x') - f(x)| = |(\widehat{f}(x') - f(x')) + (f(x') - f(x))|.$$

EXAM SUMMARY AND TACTICS

The final exam will last three hours and consist of nine questions, worth 10 marks each. You are allowed to use pens, pencils, erasers, and ingenuity.

Although they have the same weight, you will probably find some questions easier than others. Roughly six questions will be similar to topics worked in assignments, lectures, or quizzes. Three questions may be somewhat less similar. You will have the option of leaving any question blank (or writing "I don't know how to answer this question") and getting 20% (or 2/10) for that question. You will receive part marks on a prove/disprove question if you choose the wrong option but present the appropriate proof structure for that option.

The exam is comprehensive, although there will be more weight on the final part of the course that hasn't been evaluated with an assignmnt or midterm. You should be familiar with:

- logic, precise notation for sentences, Venn diagrams

- proof techniques, structured proof format

- Big Oh (and Omega, Theta)
- binary numbers, floating point representation, error

Exam preparation

I recommend that you review the following material in (roughly) this order of priority:

1. Assignments 1–4, plus their posted solutions
2. Lecture summaries
3. Midterm solutions
4. Quiz solutions

Office hours

My remaining office hours are Thursday 9pm–? in BA3222, and Monday 2–4:30 and 5:30–8pm in BA3222.

Tactics

The standard platitudes apply: try to be as well-rested as possible, have a source of caffeine (if you use it) handy. Also

- Read every exam question, since you may find some easier than others.
- Be sure you understand what you're being asked to do before you begin writing. Ask me or the other invigilator if you have questions, and we will try to prove a fair answer.
- Write the outline of a proof even if there are steps you cannot fill in. Indicate which steps you cannot fill in, rather than writing something you don't believe. Specify things you assume without proof.
- Use the space provided on the exam paper as a guide for length. The backs of the pages are provided for overflow.

Good luck.

Notes

[1] Reflecting the limit, t, on the number of digits.