CSC165, Summer 2005, Assignment 4
Sample solution

Danny Heap

All four questions received equal weight, although they were marked out of different amounts. I added 17% to all assignments, to compensate for some uniform errors in grading (for example, leaving a question blank did not earn 20%, as it should, and a mark was erroneously deducted for using methods from `Math`).

1. Complete the methods `toBiNeg(int n)`, `toInt(String s)`, and `compare(String bn1, String bn2)` from `BiNegUtil.java` (on the web page). Do not import any packages, and do not convert `bn1` or `bn2` to integers in `compare`.

```
public class BiNegUtil  {

  /**
   * toBinNeg converts int n to base -2 representation
   * @param n an integer to convert to base -2 representation
   * @return s a non-empty binary string (composed of 1's and 0's).
   *         Let s.length() = k.
   *         n = sum_{i=0}^{k-1} (-2)^i x Integer.parseInt(s[(k-1)-i])
   *         If k > 1, then s[0] == 1.
   */
  public static String toBiNeg(int n) {
    String s = "";

    if (n == 0) {
      s = "0";
    }
    else {
      while (n != 0) {
        s = Math.abs(n % -2) + s; // next bit of s

        //  bn---b1b0 / -2.0 = bn---b1 + b0/-2.0 (possibly negative)
        // so round up to get bn---b1
        n = (int) Math.ceil(n / -2.0);
       }
    }
    return s;
  }

  /**
   * toInt converts a base -2 representation of an integer to an int
   * @param s a non-empty binary string (composed of 1's and 0's).
   *         Let s.length() = k.
   *         If k > 1, then s[0] == 1.
   * @return n an int such that
   *         n = sum_{i=0}^{k-1} (-2)^i x Integer.parseInt(s[(k-1)-i])
   */
  public static int toInt(String s) {
    int n = 0;
    int twoPower = 1;
```

```
  for (int i = 0; i < s.length(); i++) {
    n += Integer.parseInt(s.substring(s.length()-1)) * twoPower;
    twoPower *= -2;
    s = s.substring(0,s.length()-2);
  }

  return n;
}

/**
 * compare returns negative, 0, or positive depending on whether bn1 < bn2
 *         bn1 == bn2, or bn1 > bn2
 * @param bn1 an integer represented as a base -2 String
 * @param bn2 an integer represented as a base -2 String
 * @return n an integer which is negative if bn1 < bn2, 0 if bn1 == bn2
 *           or positive if bn1 == bn2
 */
public static int compare(String bn1, String bn2) {
  int n = 0, twoPower = 1;

  // n holds the difference between bn1 and bn2
  for (int i = 0; i < bn1.length(); ++i) {
    if (bn2.length() > i) {
      n +=
        (Integer.parseInt(bn1.substring(bn1.length() - 1 - i, bn1.length() - i))
        - Integer.parseInt(bn2.substring(bn2.length() - 1 - i, bn2.length() - i)))
        * twoPower;
    }
    else {
      n +=
        Integer.parseInt(bn1.substring(bn1.length() - 1 - i, bn1.length() - i))
        * twoPower;
    }
    twoPower *= -2;
  }

  // if bn2 has more bits, treat the remaining bits of bn1 as zeros
  for (int i = bn1.length(); i < bn2.length(); ++i) {
    n -= twoPower
      * Integer.parseInt(bn2.substring(bn2.length() - 1 - i, bn2.length() - i));
    twoPower *= -2;
  }
  return n;
}

/**
 * mult returns the product of integers m and n
 * @param m an integer
 * @param n an integer
 * @return mn, the product of m and n
 * // precondition: m and n are integers
 * // postcondition: integer product mn is returned
 */
public static int mult(int m, int n) {
  int x = m, y = n, z = 0;
  // loop condition: z = mn - xy
  while (x != 0) {
    if (x % 2 != 0) {
      if (x * m > 0) {
        z = z - y;
      }
      else {
        z = z + y;
      }
```

```
      }
      x = (int)(Math.floor(x / -2.0));
      y = y * 2;
    }
    // postcondition z = mn
    return z;
  }
}
```

MARKING SCHEME: Marked out of 18, but weighted out of 25, so calculate $25x/18$. 5 marks for `toBiNeg(int n)`, 6 marks for `toInt(String s)`, and 7 marks for `compare(String bn1, String bn2)`.

2. Consider the base -2 representation, where $b_i \in \{0,1\}$, and

$$(b_n b_{n-1} \cdots b_1 b_0)_{-2} = \sum_{i=0}^{n} b_i(-2)^i.$$

(a) How many positive integers can be written in an n-digit base -2 representation (including representations with leading zeros on the left)? How many negative numbers can be written in an n-digit base -2 representation (including representations with leading zeros on the left). Justify your answer.

SAMPLE SOLUTION: You have at your disposal bits $(b_{n-1} \cdots b_0)_{-2} = \sum_{i=0}^{n-1} b_i(-2)^i$, where $b_i \in \{0,1\}$. There are two cases to consider, depending on whether n is even or odd:

CASE 1, n IS ODD: In this case $n-1$ is even. There are 2^0 positive base -2 numbers of the form $(0\cdots 01)_{-2}$. In general there are 2^{2k} positive base -2 numbers of the form $(0\cdots 01b_{2k-1}\cdots b_0)_{-2}$, where $0 \leq 2k \leq n-1$. We know these numbers are positive because $1 \times (-2)^{2k}$ dominates all the other terms. Summing this geometric series gives us:

$$\sum_{k=0}^{(n-1)/2} 2^{2k} = \sum_{k=0}^{(n-1)/2} 4^k = \frac{4^{(n+1)/2} - 1}{3}$$

...positive numbers.

Similarly, there are 2^1 negative base -2 numbers of the form $(0\cdots 01b_0)_{-2}$. In general there are 2^{2k+1} negative base -2 numbers of the form $(0\cdots 01b_{2k}\cdots b_0)_{-2}$, where $0 \leq 2k \leq n-3$. Summing this geometric series give us

$$\sum_{k=0}^{(n-3)/2} 2^{2k+1} = 2\sum_{k=0}^{(n-3)/2} 4^k = 2\frac{4^{(n-1)/2} - 1}{3}$$

...negative numbers.

CASE 2, n IS EVEN: In this case $n-1$ is odd, so we must sum the geometric series with different upper bounds. There are 2^{2k+1} negative base -2 numbers of the form $(0\cdots 1b_{2k}\cdots b_0)_{-2}$, where $0 \leq 2k \leq n-2$, so we sum the geometric series to get

$$\sum_{k=0}^{(n-2)/2} 2^{2k+1} = 2\sum_{k=0}^{(n-2)/2} 4^k = 2\frac{4^{n/2} - 1}{3}$$

...negative numbers.

There are 2^{2k} positive base -2 numbers of the form $(0\cdots 01b_{2k-1}\cdots b_0)_{-2}$, where $0 \leq 2k \leq n-2$, so summing the geometric series gives us

$$\sum_{k=0}^{(n-2)/2} 2^{2k} = \sum_{k=0}^{(n-2)/2} 4^k = \frac{4^{n/2} - 1}{3}$$

(b) For natural number k, what is the minimum number of digits needed to represent k in base -2 representation? Justify your answer.

SAMPLE SOLUTION: First observe that all the base -2 numbers with an even number of digits are negative, so we'll need k to be represented with $2j+1$ digits, for some natural number j. The largest number that can be represented with $2j+1$ digits is $(101\cdots01\cdots01)_{-2}$, which is $\sum_{i=0}^{j}(-2)^{2j} = \sum_{i=0}^{j} 4^j = (4^{j+1}-1)/3$. The smallest number (still positive, since it is dominated by $1\times(-2)^{2j}$) that can be represented with $2j+1$ digits is $(110\cdots10\cdots10)_{-2} = (-2)^{2j} + \sum_{i=0}^{j-1}(-2)^{2i+1} = 4^j - 2\sum_{i=0}^{j-1}4^i = 4^j - 2(4^j-1)/3 = (4^j+2)/3$. This means that:

$$\begin{aligned} \frac{4^{j+1}-1}{3} &\geq k \geq \frac{4^j+2}{3} \\ \Rightarrow 4^{j+1}-1 &\geq 3k \geq 4^j+2 \\ \Rightarrow \lfloor\log_4(4^{j+1}-1)\rfloor &\geq \lfloor\log_4(3k)\rfloor \geq \lfloor\log_4(4^j+2)\rfloor \\ j &\geq \lfloor\log_4 3k\rfloor \geq j \end{aligned}$$

Thus k requires $2\lfloor\log_4 3k\rfloor + 1$ base -2 digits.

MARKING SCHEME: Marked out of 24, so calculate $24x/25$, to take into account the weighting. Part (a) is worth 16 marks, 4 for each combination of n being odd or even with the numbers being counted being negative or positive. Part (b) is worth 8 marks, 4 for finding a range of positive numbers that can be represented by a given number of digits, 4 for deriving an expression from this.

3. Examine the method `mult(m,n)` in `BiNegUtil.java` (on the web page). Prove or disprove that if the loop invariant is true at the beginning of a loop iteration, then it is true at the end of a loop iteration. Your proof should be in the structured proof format from class.

SAMPLE SOLUTION: The statement is false. First I put the statement into precise form, and then prove its negation.

The statement to be disproved is: $\forall x', y', z', x'', y'', z'', m, n \in \mathbb{Z}$, if $z' = mn - x'y'$, and (x', y', z') are the values of (x, y, z) before the loop iteration and (x'', y'', z'') are the values of (x, y, z) after the loop iteration, then $z'' = mn - x''y''$.

The negation of this statement is: $\exists x', y', z', x'', y'', z'', m, n \in \mathbb{Z}$, $z' = mn - x'y' \wedge (x', y', z')$ are the values of (x, y, z) before the loop iteration, (x'', y'', z'') are the values of (x, y, z) after the loop iteration $\wedge\ z'' \neq mn - x''y''$. I now prove the negation.

Let $x' = m = 3$, $n = y' = 2$, $z' = 0$, $x'' = -2$, $y'' = 4$, and $z'' = 2$, and assume (x', y', z') are the values of (x, y, z) before the loop iteration.

Then $x, y, z, x', y', z', m, n$ are integers, and $mn - xy = 6 - 6 = 0 = z'$.
Since $x \neq 0$ and $x = 3$ is odd and $x * m = 9 > 0$, the program sets the new value of z to $0 - y = -2$, so $z'' = -2$ is the value of z after this iteration.
The program sets x'' to $\lfloor 3/(-2)\rfloor = -2$, and y'' to $2 * y' = 4$. So $(x'', y'', z'') = (-2, 4, -2)$ are the values of (x, y, z) after one iteration.
So $mn - x''y'' = 6 - (-2)(4) = 14 \neq z'' =$.
Thus $z' = mn - x'y' \wedge (x', y', z')$ are the values of (x, y, z) before the loop iteration $\wedge$ (x'', y'', z'') are the values of (x, y, z) after the loop iteration, and $z'' \neq mn - x''y''$.

Since $x, y, z, x', y', z', m, n$ are integers, $\exists x', y', z', x'', y'', z'', m, n \in \mathbb{N}$, $z' = mn - x'y' \wedge (x'', y'', z'')$ are the values of (x, y, z) after one iteration of the loop $\wedge\ z'' \neq mn - x''y''$.

MARKING SCHEME: Marked out of 8, so calculate $25x/8$ to take account of the weighting. No marks for proving the invariant true.

4. Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and let

$$O(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geq 0} | \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \to g(n) \leq cf(n)\}$$

Prove or disprove (in structured proof form) the following claims:

(a) Suppose $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Then $g \in O(f) \Rightarrow g(n^3) \notin O(f)$

SAMPLE SOLUTION: The claim is false. I prove the negation. Let $F = \{h : \mathbb{N} \to \mathbb{R}^{\geq 0}\}$.

$$\exists f \in F, \exists g \in F, g \in O(f) \wedge g(n^3) \in O(f).$$

Let $f(n) = g(n) = 1$ for $n \in \mathbb{N}$. (constant functions).

Then $f, g \in F$, since $1 \in \mathbb{R}^+$. Let $c = 1$ and let $B = 0$.

Then $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Assume $n \geq B = 0$.

Then $g(n) = 1 \leq cf(n) = 1$. (by choice of f, g, and c).
Also $g(n^3) = 1 \leq cf(n)$. (by choice of f, g, and c).

So $n \geq B \Rightarrow g(n) \leq cf(n)$.
Also $n \geq B \Rightarrow g(n^3) \leq cf(n)$
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
Also, since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow g(n^3) \leq cf(n)$.

Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
So $g \in O(f)$. (definition of $O(f)$).
Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n^3) \leq cf(n)$.
So $g(n^3) \in O(f)$. (definition of $O(f)$).

Since f and g are in F, $\exists f \in F, \exists G \in F, g \in O(f) \wedge g(n^3) \in O(f)$.

(b) Suppose $f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Then $(g \in O(f) \wedge h \in O(f)) \Rightarrow \max(g, h) \in O(f)$.

SAMPLE SOLUTION: The statement is true. Define (for convenience) $F = \{h : \mathbb{N} \to \mathbb{R}^+\}$.

Let $f, g \in F$. Assume $g \in O(f) \wedge h \in O(f)$.

Then $g \in O(f)$.
So $\exists c \in \mathbb{R}^+, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$. (definition of $g \in O(f)$.).
Let $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_1 \Rightarrow g(n) \leq c_1 f(n)$.
Then $h \in O(f)$.
So $\exists c \in \mathbb{R}^+, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow h(n) \leq cf(n)$. (definition of $g \in O(f)$).
Let $c_2 \in \mathbb{R}^+$ and $B_2 \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_2 \Rightarrow h(n) \leq c_2 f(n)$.
Let $c = c_1 + c_2$. Let $B = \max(B_1, B_2)$.

Then $c \in \mathbb{R}^+$. (sum of positive real numbers is a positive real number).
Then $B \in \mathbb{N}$. (sum of natural numbers is a natural number).
Let $n \in \mathbb{N}$. Assume $n \geq B$.

Then $n \geq B_1$. (Since $B = \max(B_1, B_2)$).
So $g(n) \leq c_1 f(n)$. (shown above).
So $g(n) \leq (c_1 + c_2) f(n)$. (since $c_2 f(n)$ is non-negative).
Then $n \geq B_2$. (since $B = \max(B_1, B_2)$).
So $h(n) \leq c_2 f(n)$. (shown above).
So $h(n) \leq (c_1 + c_2) f(n)$ (since $c_1 f(n)$ is non-negative).
So $\max(g(n), h(n)) \leq (c_1 + c_2) f(n) = cf(n)$. (by the definition of max and choice of c).

Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow \max(g(n), h(n)) \leq cf(n)$.

Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow \max(g(n), h(n)) \leq cf(n)$.
So $\max(g, h) \in O(f)$. (definition of $O(f)$).
Thus $(g \in O(f) \wedge h \in O(f)) \Rightarrow \max(g, h) \in O(f)$.

Since f, g, and h are arbitrary elements of F, $\forall f \in F, \forall g \in F, \forall h \in F, (g \in O(f) \wedge h \in O(f)) \Rightarrow \max(g, h) \in O(f)$.

(c) Suppose $f, f', g, g' : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, and $f \circ g(n) = f(g(n))$, $f' \circ g'(n) = f'(g'(n))$. Then $(f \in O(f') \wedge g \in O(g')) \Rightarrow f \circ g \in O(f' \circ g')$.

Sample solution: The claim is false. Let $F = \{h : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$. I will prove the negation:

$$\exists f \in F, \exists f' \in F, \exists g \in F, \exists g' \in F, f \in O(f') \wedge g \in O(g') \wedge f \circ g \notin O(f' \circ g')$$

Sample solution: The statement is false. I prove the negation (where $F = \{h : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$):

$$\exists f, f', g, g' \in F, f \in O(f') \wedge g \in O(g') \wedge f \circ g \notin O(f' \circ g')$$

Let $f(n) = 2^n = f'(n)$. Let $g(n) = 2n$. Let $g'(n) = n$.

Then $f, f', g, g' \in F$. (Their range if $\mathbb{N} \subset \mathbb{R}^{\geq 0}$).
Let $c_1 = 1$. Let $B_1 = 0$.
Then $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Assume $n \geq B_1 = 0$.
Then $f(n) = 2^n \leq c_1 f'(n) = 2^n$. (by choice of f, f', and c).
So $n \geq B_1 \Rightarrow f(n) \leq cf'(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $n \geq B_1 \Rightarrow f(n) \leq cf'(n)$.
Since $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cf'(n)$.
So $f \in O(f')$. (definition).
Let $c_2 = 2$. Let $B_2 = 0$.
Then $c_2 \in \mathbb{R}^+$ and $B_2 \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Assume $n \geq B_2$.
Then $g(n) = 2n \leq 2n = c_2 g'(n)$. (by choice of g, g', and c_2).
So $n \geq B_2 \Rightarrow g(n) \leq c_2 g'(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B_2 \Rightarrow g(n) \leq c_2 g'(n)$.
Since $c_2 \in \mathbb{R}^+$ and $B_2 \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow g(n) \leq cg'(n)$.
So $g \in O(g')$. (definition).
Let $c \in \mathbb{R}^+$. Let $B \in \mathbb{N}$.
Let $n = \max(\lceil [\log_2 c] + 1 \rceil, B)$.
Then $n \in \mathbb{N}$. (maximum of two natural numbers).
Then $n \geq B$. (definition of max).
Then $2^n \geq 2^{(\log_2 c)+1} = 2c > c$. (by choice of n).
So $2^n \times 2^n > c2^n$. (multiply the inequality by 2^n).
So $f(g(n)) = f(2n) = 2^{2n} = 2n \times 2n > c2n = c2^{g'(n)} = f'(g'(n))$. (by the previous inequality and choice of f, f', g, g').
So $n \geq B$ and $f(g(n)) > f'(g'(n))$.
Since $n \in \mathbb{N}$, $\exists n \in \mathbb{N}$, $n \geq B \wedge f(g(n)) > f'(g'(n))$.
Since c is an arbitrary element of $\mathbb{R}^+$ and B is an arbitrary element of $\mathbb{N}$, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(g(n)) > f'(g'(n))$
So $f \circ g \notin O(f' \circ g')$. (definition of $O(f'g')$).

Since f, f', g, g' are elements of $\mathbb{F}$, $\exists f, f', g, g' \in F, f \in O(f') \wedge g \in O(g') \wedge f \circ g \notin O(f' \circ g')$

(d) Suppose $f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $gh(n) = g(n)h(n)$. Then $(g \in O(f) \wedge h \in O(f)) \to gh \in O(f)$.

SAMPLE SOLUTION: The statement is false. Let $F = \{h : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$. I will prove the negation:

$$\exists f, g, h \in F, g \in O(f) \wedge h \in O(f) \wedge gh \notin O(f)$$

Let $f(n) = g(n) = h(n) = n$.

Then $f, g, h \in F$. (since they map $\mathbb{N}$ to $\mathbb{N} \subset \mathbb{R}^{\geq 0}$).
Let $c = 1$. Let $B = 0$.
Then $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Assume $n \geq B = 0$.
Then $g(n) = n \leq n = cf(n)$. (by the choice of f, g, and c).
Then $h(n) = n \leq n = cf(n)$. (by the choice of f, h, and c).
So $n \geq B \Rightarrow g(n) \leq cf(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
So $n \geq B \Rightarrow h(n) \leq cf(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow h(n) \leq cf(n)$.
Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
So $g \in O(f)$. (definition)
Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}R^+, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow h(n) \leq cf(n)$.
So $h \in O(f)$. (definition).
Let $c \in \mathbb{R}^+$. Let $B \in \mathbb{N}$.
Let $n = \max(B, \lceil c \rceil + 1)$.
Then $n \in \mathbb{N}$. (maximum of natural numbers, since c is positive, its ceiling is a natural number, and the sum of natural numbers is a natural number).
Then $n \geq B$ (definition of max).
Then $g(n)h(n) = n^2 = n \times n > cn$ (since $n \geq c + 1 > c$).
So $g(n)h(n) > cf(n)$.
Since $n \in \mathbb{N}$, $\exists n \in \mathbb{N}, n \geq B \wedge g(n)h(n) > cf(n)$.
Since c is an arbitrary element of $\mathbb{R}^+$ and B is an arbitrary element of $\mathbb{N}$, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge g(n)h(n) > cf(n)$.
So $g, h \notin O(f)$. (definition of $O(f)$, negated).

Since $f, g, h \in F, \exists f, g, h \in F, g \in O(f) \wedge h \in O(f) \wedge gh \notin O(f)$.

(e) Suppose $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $f'(n) = \lfloor f(n) \rfloor$, $g'(n) = \lfloor g(n) \rfloor$. Then $g \in O(f) \to g' \in O(f')$.

SAMPLE SOLUTION: The statement is false. Let $F = \{h : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$. I prove the negation:

$$\exists f, g, f', g' \in F, f'(n) = \lfloor f(n) \rfloor, g'(n) = \lfloor g(n) \rfloor \wedge g \in O(f) \wedge g' \notin O(f')$$

Let $f(n) = 0.5$, $f'(n) = \lfloor f(n) \rfloor = 0$, $g(n) = 1$, $g'(n) = \lfloor g(n) \rfloor = 1$. (constant functions).

Then $f, f', g, g' \in F$ (they map natural numbers to non-negative real numbers).
Let $c = 2$. Let $B = 0$.
Let $n \in \mathbb{N}$. Assume $n \geq B = 0$.
Then $g(n) = 1 \leq 1 = cf(n)$. (by choice of f, g, c).
So $n \geq B \Rightarrow g(n) \leq cf(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq f(n)$.
Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
So $g \in O(f)$. (definition).
Let $c' \in \mathbb{R}^+$. Let $B' \in \mathbb{N}$

Let $n = B'$.

Then $n \in \mathbb{N}$ (since B' is).
Also $n \geq B'$ (since $n = B'$).
Also $1/c' > 0$ (since reciprocal of a positive number is positive).
So $1 > c'0$. (multiplying both sides by positive c').
So $g'(n) = 1 > c'0 = c'f'(n)$.

Since $n \in \mathbb{N}$, $\exists n \in \mathbb{N}, n \geq B' \wedge g'(n) > c'f'(n)$.

Since c' is an arbitrary element of $\mathbb{R}^+$ and B' is an arbitrary element of $\mathbb{N}$, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge g'(n) > cf'(n)$.
So $g' \notin O(f')$. (definition of $O(f')$, negated).

Since $f, f', g, g' \in F$, $\exists f, f', g, g' \in F, f'(n) = \lfloor f(n) \rfloor \wedge g'(n) = \lfloor g(n) \rfloor \wedge g \in O(f) \wedge g' \notin O(f')$.

Marking scheme: Each part was marked out of 10, so this question was marked out of 50, calculate $25x/50$ to account for the weighting. No marks for proving a false claim true, or vice versa. Two marks for stating (correctly) the truth or falsity of the statement and quantifying the functions, and a mark for the conclusion. Four marks for quantifying constants and/or constructing new constants. Three marks for showing that the required inequality holds.