CSC165, Summer 2005, Assignment 2
Sample solution

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1. Let $\mathbb{N}$ be the natural numbers $\{0, 1, 2, \ldots\}$, $\mathbb{Z}$ be the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and $\mathbb{R}$ be the real numbers. For $x \in \mathbb{R}$, define $r(x)$ as: $\exists m \in \mathbb{N}, \exists n \in \mathbb{N}, (n > 0) \wedge (x = m/n)$. You may assume $\neg r(\sqrt{2})$.

   Using our structured proof form, prove or disprove the following:

   (a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (r(x) \wedge r(y)) \Rightarrow r(x + y)$.

   Sample solution: The statement is true.

   Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.
   Assume $r(x) \wedge r(y)$.
   Then $\exists m_x, n_x \in \mathbb{N}$, $n_x > 0 \wedge x = m_x/n_x$. (by definition of $r(x)$).
   Then $\exists m_y, n_y \in \mathbb{N}$, $n_y > 0 \wedge y = m_y/n_y$. (by definition of $r(y)$).
   Let $m_{x+y} = n_y m_x + n_x m_y$. Let $n_{x+y} = n_x n_y$.
   Then $m_{x+y} \in \mathbb{N}$. (since natural numbers are closed under multiplication and addition).
   Then $n_{x+y} \in \mathbb{N}$ and $n_{x+y} \neq 0$. (since natural numbers are closed under multiplication, and the product of non-zero natural numbers is not zero).
   Also $x + y = m_{x+y}/n_{x+y}$. (definition of addition of fractions).
   Hence $\exists m \in \mathbb{N}$, $\exists n \in \mathbb{N}$ $n > 0$ and $x + y = m/n$.
   Thus $r(x + y)$. (by definition of $r(x + y)$).
   So $(r(x) \wedge r(y)) \Rightarrow r(x + y)$.
   Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $(r(x) \wedge r(y)) \Rightarrow r(x + y)$.

   (b) The converse of (a)

   Sample solution: The statement is false.

   Let $x = \sqrt{2}$. Let $y = 2 - \sqrt{2}$.
   Then $x \in \mathbb{R}$. (real numbers include roots of positive reals).
   Then $y \in \mathbb{R}$. (real numbers are closed under subtraction).
   Let $m = 2$. Let $n = 1$.
   Then $m \in \mathbb{N}$.
   Then $n \in \mathbb{N}$ and $n \neq 0$.
   Also $x + y = 2/1$. (since $\sqrt{2} + 2 - \sqrt{2} = 2$).
   Hence $\exists m \in \mathbb{N}$ and $\exists n \in \mathbb{N}$, $n \neq 0$ and $x + y = m/n$.
   So $r(x + y)$ and $\neg r(x)$. (by definition of $r(x + y)$, and given assumption that $\neg r(\sqrt{2})$).
   Thus $r(x + y) \wedge \neg(r(x) \wedge r(y))$. (since $\neg r(x)$ implies $\neg r(x) \vee \neg r(y)$).
   Thus $r(x + y) \not\Rightarrow (r(x) \wedge r(y))$. (by negation of implication).

Thus $\exists x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $r(x+y) \not\Leftarrow (r(x) \wedge r(y))$.

(c) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (r(x) \wedge r(y)) \Rightarrow r(xy)$.

Sample solution: The statement is true.

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.

Assume $r(x) \wedge r(y)$

Then $\exists m_x, n_x \in \mathbb{N}$, $n_x > 0 \wedge x = m_x/n_x$. (by definition of $r(x)$).
Then $\exists m_y, n_y \in \mathbb{N}$, $n_y > 0 \wedge y = m_y/n_y$. (by definition of $r(y)$).
Let $m_{xy} = m_x m_y$. Let $n_{xy} = n_x n_y$

Then $m_{xy} \in \mathbb{N}$. (natural numbers are closed under multiplication).
Then $n_{xy} \in \mathbb{N}$ and $n_{xy} \neq 0$. (natural numbers are closed under multiplication and the product of non-zero natural numbers is non-zero).
Also, $xy = m_{xy}/n_{xy}$.

Hence $\exists m \in \mathbb{N}$, $\exists n \in \mathbb{N}$, $n \neq 0$ and $xy = m/n$.
Thus $r(xy)$. (by definition of $r(xy)$).

So $(r(x) \wedge r(y)) \Rightarrow r(xy)$.

Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $(r(x) \wedge r(y)) \Rightarrow r(xy)$.

(d) The converse of (c)

Sample solution: The statement is false

Let $x = \sqrt{2}$. Let $y = \sqrt{2}$.

Then $x \in \mathbb{R}$. (real numbers include roots of positive reals).
Then $y \in \mathbb{R}$. (real numbers include roots of positive reals).
Let $m = 2$. Let $n = 1$.

Then $m \in \mathbb{N}$.
Then $n \in \mathbb{N}$ and $n \neq 0$.
Also $xy = m/n$. (since $\sqrt{2}^2 = 2$).

Hence $\exists m \in \mathbb{N}$, $n \in \mathbb{N}$, $n \neq 0$ and $xy = m/n$.
So $r(xy)$ and $\neg r(x)$. (definition of $r(xy)$ and given assumption that $\neg r(\sqrt{2})$).
So $r(xy)$ and $\neg(r(x) \wedge r(y))$. (at least one of $r(x), r(y)$ is false).
So $r(xy) \not\Rightarrow (r(x) \wedge r(y))$. (negation of implication)

Thus $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}$, $r(xy) \not\Rightarrow (r(x) \wedge r(y))$.

2. For $x \in \mathbb{R}$, define $|x|$ by

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Using our structured proof form, prove or disprove the following. You may assume that if $t > 0$, then $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow tx > ty$.

(a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x||y| = |xy|$.

Sample solution: The statement is true.

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.

Case 1, $x < 0$ and $y < 0$.

Then $|x| = -x$ and $|y| = -y$. (definition of $|x|$ and $|y|$).
So $|x||y| = (-x)(-y) = xy$. (since $(-1)^2 = 1$).
Also $xy > 0$. (product of negative numbers is positive).
So $xy = |xy|$. (definition of $|xy|$ when $xy \geq 0$).
Hence $|x||y| = |xy|$.

Case 2, $x < 0$ and $y \geq 0$.

Then $|x| = -x$ and $|y| = y$. (definition of $|x|$ and $|y|$).
So $|x||y| = -xy$.
Also $xy \leq 0$. (product of a negative and a non-negative number is either 0 or negative).
So either $|xy| = -xy$ (by the definition of $|xy|$ when $xy < 0$), or $|xy| = xy = -xy$ (by the definition of $|xy|$ when $xy = 0$).
Thus $|x||y| = |xy|$.

Case 3, $x \geq 0$ and $y < 0$.

Then $|x| = x$ and $|y| = -y$. (definition of $|x|$ and $|y|$).
So $|x||y| = -xy$.
Also $xy \leq 0$. (product of a non-negative number with a negative number is non-positive).
So either $|xy| = -xy$ (by the definition of $|xy|$ for $xy < 0$) or $|xy| = xy = -xy$ (by the definition of $|xy|$ for $xy = 0$).
So $|x||y| = |xy|$.

Case 4, $x \geq 0$ and $y \geq 0$.

Then $|x| = x$ and $|y| = y$. (definition of $|x|$ and $|y|$).
Also $xy \geq 0$. (product of non-negative numbers is non-negative).
So $|xy| = xy$. (definition of $|xy|$).
Thus $|x||y| = |xy|$.

In each case $|x||y| = |xy|$, and these cover all possibilities. So $|x||y| = |xy|$.

Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x||y| = |xy|$.

(b) $\forall x_1 \in \mathbb{R}, \forall x_2 \in \mathbb{R}, \forall y_1 \in \mathbb{R}, \forall y_2 \in \mathbb{R}, |x_1| > |x_2| \wedge |y_1| > |y_2| \Rightarrow |x_1 y_1| > |x_2 y_2|$.

Sample solution: The statement is true.

Let $x_1 \in \mathbb{R}$. Let $x_2 \in \mathbb{R}$. Let $y_1 \in \mathbb{R}$. Let $y_2 \in \mathbb{R}$.
Assume $|x_1| > |x_2| \wedge |y_1| > |y_2|$.

Let $t = |y_1|$.

Then $t \in \mathbb{R}$. (definition of $|y_1|$).
Then $t > |y_2|$. (Since $t = |y_1| > |y_2|$, by assumption).
Also $|y_2| \geq 0$. (Since either $|y_2| = -y2 > 0$ if y_2 is negative, or $|y_2| = y_2 \geq 0$ if y_2 is non-negative, by definition of $|y_2|$.
So $t > 0$. (Since $t > |y - 2| \geq 0$).
Then $t|x_1| > t|x_2|$. (Since $|x_1| > |x_2|$, by assumption, and the result we are allowed to assume for this question).

So $|x_1||y_1| > |x_2||y_1|$. (by construction of t and commutativity of multiplication)
Let $t = |x_2|$.

Then $t \in \mathbb{R}$. (by definition of $|x_2|$.
Also $t \geq 0$. (since either $|x_2| = -x_2 > 0$ if x_2 is negative, or $|x_2| = x_2 \geq 0$ if x_2 is non-negative, by definition of $|x_2|$.).
So $t|y_1| \geq t|y_2|$. (since either $t|y_1| > t|y_2|$, by the result we are allowed to assume for this question when $t > 0$, or $t|y_1| = t|y_2|$ when $t = 0$).

So $|x_2||y_1| \geq |x_2||y_2|$. (by definition of t)
So $|x_1||y_1| > |x_2||y_2|$. (Since $|x_1||y_1| > |x_2||y_1|$ and $|x_2||y_1| \geq |x_2||y_2|$.
So $|x_1 y_1| > |x_2 y_2|$. (Since $|x_1||y_1| = |x_1 y_1|$ and $|x_2||y_2| = |x_2 y_2|$, by part (a)).

So $|x_1| > |x_2| \wedge |y_1| > |y_2| \Rightarrow |x_1 y_1| > |x_2 y_2|$.

Since x_1, x_2, y_1, y_2 are arbitrary elements of $\mathbb{R}$, $\forall x_1 \in \mathbb{R}$, $\forall y_1 \in \mathbb{R}$, $\forall y_2 \in \mathbb{R}$, $(|x_1| > |x_2| \wedge |y_1| > |y_2|) \Rightarrow |x_1 y_1| > |x_2 y_2|$.

3. Let $\mathbb{R}^+$ be the set of positive real numbers. Use our structured proof form to prove or disprove:

(a)
$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

Sample solution: The statement is (strangely) true.

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.

Case 1, $x = y$

Let $\delta = 17$.

Then $\delta \in \mathbb{R}^+$

Let $\epsilon \in \mathbb{R}^+$.

Then $|x^2 - y^2| = 0 < \epsilon$. (since $x = y$ and ϵ is positive).

So $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$. (since the consequent is true, the entire implication is true).

Since ϵ is an arbitrary element of $\mathbb{R}^+$, $\forall \epsilon \in \mathbb{R}^+$, $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+$, $\forall \epsilon \in \mathbb{R}^+$, $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

Case 2, $x \neq y$.

Then $|x - y| > 0$. (since $x - y \neq 0$ and either $|x - y| = x - y$, if $x > y$, or $|x - y| = y - x$, if $y > x$).

Let $\delta = |x - y|/2$.

Then $\delta \in \mathbb{R}^+$. (since δ is half of a positive number).

Also, $|x - y| > \delta$. (since $|x - y| - \delta = \delta > 0$).

Let $\epsilon \in \mathbb{R}$.

Then $\neg(|x - y| < \delta)$. (since $|x - y| > \delta$).

So $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$. (since a false antecedent implies anything).

Since ϵ is an arbitrary element of $\mathbb{R}^+$, $\forall \epsilon \in \mathbb{R}^+$, $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+$, $\forall \epsilon \in \mathbb{R}^+$, $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

In either case $\exists \delta \in \mathbb{R}^+$, $\forall \epsilon \in \mathbb{R}^+$, $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$, and these cases cover all possibilities.

So $\exists \delta \in \mathbb{R}^+$, $\forall \epsilon \in \mathbb{R}^+$, $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $\exists \delta \in \mathbb{R}^+$, $\forall \epsilon \in \mathbb{R}^+$, $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

(b)
$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

Sample solution: The claim is true.

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$. Let $\epsilon \in \mathbb{R}$.

Case 1, $x^2 - y^2 = 0$.

Let $\delta = 1$.

Then $\delta \in \mathbb{R}^+$.

Assume $|x - y| < \delta$.

Then $|x^2 - y^2| < \epsilon$. (since $\epsilon \in \mathbb{R}^+$)

So $|x - y| < \delta| \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$

Case 2, $x^2 - y^2 \neq 0$

Then $(x - y)(x + y) \neq 0$. (by factoring quadratic).

So $(x - y) \neq 0$ and $(x + y) \neq 0$. (non-zero product has no zero factors).

Let $\delta = \epsilon/(2|x + y|)$.

Then $\delta \in \mathbb{R}^+$. (since it ratio of positive numbers).

Assume $|x - y| < \delta$.

Then $|x^2 - y^2| = |x - y||x + y|$. (by part 2a).

So $|x^2 - y^2| < \delta|x + y|$. (by assumption that $|x - y| < \delta$).

So $\delta|x + y| = [\epsilon/(2|x + y|)] \times |x + y|$. (by construction of δ).

So $|x^2 - y^2| < \epsilon/2 < \epsilon$. (Since $\epsilon - \epsilon/2 = \epsilon/2 > 0$).

So $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

In either case $\exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$, and this covers all possibilities.

Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$

(c)

$$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in [-1, 1], \forall y \in [-1, 1], |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

Sample solution: The claim is true.

Let $\epsilon \in \mathbb{R}^+$.

Let $\delta = \epsilon/2$.

Then $\delta \in \mathbb{R}^+$. (since $\epsilon \in \mathbb{R}^+$ means $\epsilon/2$ is a ratio of positive real numbers).

Let $x \in [-1, 1]$. Let $y \in [-1, 1]$.

Assume $|x - y| < \delta$.

Then $(x + y) \in [-2, 2]$. (Taking the maximum and minimum sums).

So $|x + y| \leq 2$. (Taking the maximum absolute value).

So $|x - y||x + y| \leq |x - y|2$.

So $|x^2 - y^2| \leq |x - y|2 < 2\delta$. (By the assumption that $|x - y| < \delta$).

So $|x^2 - y^2| < 2\epsilon/2 = \epsilon$. (By the construction of ϵ).

So $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

Since x and y are arbitrary elements of $[-1, 1]$, $\forall x \in [-1, 1], \forall y \in [-1, 1], |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+, \forall x \in [-1, 1], \forall y \in [-1, 1], |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

Since ϵ is an arbitrary element of $\mathbb{R}$, $\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in [-1, 1], \forall y \in [-1, 1], |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

(d)

$$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

Sample solution: The claim is false, so I will prove its negation:

$$\exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x - y| < \delta \wedge |x^2 - y^2| \geq \epsilon.$$

Let $\epsilon = 1$.

Let $\delta \in \mathbb{R}^+$.

Let $x = 2/\delta$. Let $y = x + \delta/2$.

So $|x - y| = \delta/2 < \delta$. (since $\delta - \delta/2 = \delta/2 > 0$).

So $|x^2 - y^2| = |x - y||x + y| = \delta/2(2 \times 2/\delta + \delta/2)$ (by construction of x and y).

So $|x^2 - y^2| = 2 + \delta^2/4 > \epsilon$. (Since $\epsilon = 1$, and $\delta^2/4$ is positive).

So $|x - y| < \delta$ and $|x^2 - y^2| > \epsilon$.

So $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x - y| < \delta \wedge |x^2 - y^2| \geq \epsilon$.

Since δ is an arbitrary element of $\mathbb{R}^+$, $\forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x - y| < \delta \wedge |x^2 - y^2| \geq \epsilon$

So $\exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x - y| < \delta \wedge |x^2 - y^2| \geq \epsilon$

4. Suppose f and g are functions from $\mathbb{R}$ onto $\mathbb{R}$. Consider the following statements:

S1 $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (f(x) = f(y)) \Rightarrow (x = y)$.

S2 $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (g(x) = g(y)) \Rightarrow (x = y)$.

S3 $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (g(f(x)) = g(f(y))) \Rightarrow (x = y)$.

Does (S1∧S2) imply S3? Prove your claim.

SAMPLE SOLUTION: The claim is true.

Assume S1 ∧ S2

So S1

So S2

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.

Assume $g(f(x)) = g(f(y))$.

Let $x' = f(x)$. Let $y' = f(y)$.

Then $x' \in \mathbb{R}$ and $y' \in \mathbb{R}$. (by assumption that f and g are from $\mathbb{R}$ to $\mathbb{R}$).

So $x' = y'$. (By assumption of S2, since $g(x') = g(y')$).

So $f(x) = f(y)$. (by construction of x' and y').

So $x = y$. (By assumption of S1, since $f(x) = f(y)$).

So $g(f(x)) = g(f(y)) \Rightarrow x = y$.

Since x and y are arbitrary elements of $\mathbb{R}$, $g(f(x)) = g(f(y)) \Rightarrow x = y$.

Then S3. (by definition of S3).

Hence S1 ∧ S2 ⇒ S3.