Sample solution
CSC165, Summer 2005, Assignment 2

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1. For $x, y, z \in \mathbb{R}$ define:

   - $d(x, y, z)$ means $x/y = z$.
   - $s(x, y, z)$ means $x + y = z$.
   - $eq(x, y)$ means $x = y$.
   - $g(x, y)$ means $x > y$.
   - $\mathbb{N}(x)$ means $x \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ (the natural numbers).

   Express each of the following symbolic sentences in English. If the sentence is false, provide a counterexample. If it's true, explain why.

   (a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}, s(x, y, z)$.

   Every pair of real numbers has a sum which is also a real number. This is true, since the real numbers are closed under addition.

   (b) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, \mathbb{N}(x) \wedge \mathbb{N}(y) \wedge s(x, y, z) \Rightarrow \mathbb{N}(z)$.

   If any pair of real numbers are also natural numbers, their sum is also a natural number. True, since the natural numbers are closed under addition.

   (c) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}, d(x, y, z)$.

   For every pair of real numbers there exists a real number that is their quotient. False, for example their is no quotient $5/0$.

   (d) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, s(x, y, x)$.

   There is some real number whose sum, when added to any other real number, is the second real number. True, let $y = 0$.

   (e) $\forall y \in \mathbb{R}, \forall x \in \mathbb{R}, s(x, y, x) \Rightarrow eq(y, 0)$.

   If the sum of a pair of real numbers is equal to the first element of the pair, then the second element of the pair is zero. True, since if $x + y = x$, then (subtracting x from both sides) $y = 0$.

   (f) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, d(x, y, x) \Rightarrow eq(y, 1)$.

   If the quotient of a pair of real numbers is the first element of the pair, then the second element is 1. False, since $0/3 = 0$ provides a counter-example.

   (g) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, s(x, y, 0)$.

   Every real number has an additive inverse, that is a number which added to the original number yields zero. True, since for every $x \in \mathbb{R}$ there is $-x \in \mathbb{R}$.

   (h) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, s(x, y, 0)$.

   There is some real number that is an additive inverse for every number. False, since (for example) 7 and 5 have distinct additive inverses.

(i) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, g(x,y) \wedge d(z,x,y) \Rightarrow g(z,y)$.

For every pair of real numbers, if the first element is greater than the second, their the product of the two is greater than the second. False, since $(1,-2)$ provides a counterexample.

2. Using the predicates defined in the previous question, and $\mathbb{N} = \{0,1,2,\ldots,\}$:

(a) Translate the following into English:

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, d(x,3,y) \wedge d(z,x,x) \Rightarrow \exists w \in \mathbb{N}, d(z,3,w)$$

If a natural number is a natural multiple of 3, then its square is also a natural multiple of 3.

(b) If the sentence you translated above is false, provide a counterexample. Otherwise, write a direct proof of the implication.

Let $x, y, z \in \mathbb{N}$

Suppose $d(x,3,y)$ and $d(z,x,x)$

Then $x = 3y$ and $z = x^2$ (definition of $d(x,3,y)$ and $d(z,x,x)$).

So $z = 9y^2$ (substituting $x = 3y$ into $z = x^2$).

So $z = 3(3y^2)$.

Let $k = 3y^2$.

Then $k \in \mathbb{N}$ ($\mathbb{N}$ closed under multiplication)

Since $k \in \mathbb{N}$, there is a natural number k such that $z = 3k$.

So z is a natural multiple of 3.

So if $x = 3y$ and $z = x^2$, then z is a natural multiple of 3.

Since x, y, and z were chosen arbitrarily, any natural multiple of 3 has a square that is also a natural multiple of 3. QED.

3. Consider the following java method (if you have ANY questions about java, please feel free to ask your TA or instructor):

```
public static boolean longPredicate(boolean p, boolean q) {
 return (
        ((!p || q) && (p || !q) && !q) ||
        (!(p && !q) && !(!p && q) && p)
          );
}
```

(a) Make a literal translation of the conditional expression being returned into precise symbolic notation, using p, q, $\wedge$, $\vee$, and $\neg$, but without using $\Rightarrow$ or $\Leftrightarrow$. p and q should occur as many times in your expression as they do in the java conditional expression.

In this step, I simply replace ! by $\neg$, || by $\vee$ and && by $\wedge$:

$$((\neg p \vee q) \wedge (p \vee \neg q) \wedge \neg q) \vee (\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) \wedge p)$$

(b) Simplify your expression from the previous part, allowing yourself to use $\Rightarrow$ or $\Leftrightarrow$ where appropriate. Use p and q no more than twice each. Explain the steps you took in simplifying the expression.

I use DeMorgan's laws (DL), the distributive, identity, and idempotent properties (DP, IP, IDP) repeatedly:

$$\begin{aligned}
((\neg p \vee q) \wedge (p \vee \neg q) \wedge \neg q) \vee (\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) \wedge p) &\Leftrightarrow \text{DP, DL} \\
(((\neg p \wedge p) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg q)) \wedge \neg q) \vee ((\neg p \vee q) \wedge (p \vee \neg q) \wedge p) &\Leftrightarrow \text{(IP, DP)} \\
(((\neg p \wedge \neg q) \vee (p \wedge q)) \wedge \neg q) \vee (((\neg p \wedge p) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg q)) \wedge p) &\Leftrightarrow \text{(DP, IP)} \\
(\neg p \wedge \neg q \wedge \neg q) \vee (((\neg p \wedge \neg q) \vee (q \wedge p)) \wedge p) &\Leftrightarrow \text{(DP,IP,IDP)} \\
(\neg p \wedge \neg q) \vee (q \wedge p \wedge p) \Leftrightarrow (\neg p \wedge \neg q) \vee (p \wedge q) &\Leftrightarrow (p \Leftrightarrow q).
\end{aligned}$$

The last identity was shown in lecture.

(c) Write the negation of the previous part in precise symbolic notation, and explain its meaning in English.

Negating the last line with DeMorgan's laws yields:

$$(p \vee q) \wedge (\neg p \vee \neg q).$$

Thus at least one of p or q is true, and at least one of p or q is false. This is the EXCLUSIVE OR: exactly one of p or q is true. It is the negation of the biconditional (double conditional).

4. Define $P(x)$ as "x is pedantic," $Q(x)$ as "x is a quibbler," $R(x)$ as "x is redundant," and S is the domain of scholars. For each sentence below:

- Write the negation of the sentence in precise symbolic notation, moving the negation symbol $\neg$ as close as possible to the predicates P, Q or R as possible.
- Draw a Venn diagram showing P, Q, R and S for which the original sentence is false.

Here are the sentences:

(a) Any redundant scholar is a quibbler only if he/she is pedantic.

$$\neg(\forall s \in S, (R(s) \wedge Q(s)) \Rightarrow P(s)) \quad \Leftrightarrow \quad \exists s \in S, R(s) \wedge Q(s) \wedge \neg P(s)$$

Any Venn diagram in which some part of $R \cap Q \cap \neg P$ is shaded makes the original sentence false.

(b) Some scholar is not redundant if he/she is neither a quibbler nor non-pedantic.

$$\neg(\exists s \in S, \neg(Q(s) \vee \neg P(s)) \Rightarrow \neg R(s)) \Leftrightarrow \forall s \in S, \neg Q(s) \wedge P(s) \wedge R(s)$$

Any Venn diagram that does not shade outside of $\neg Q \cap P \cap R$ makes the original sentence false.

(c) All redundant and pedantic scholars must be quibblers.

$$\neg(\forall s \in S, R(s) \wedge P(s) \Rightarrow Q(s)) \Leftrightarrow \exists s \in S, R(s) \wedge P(s) \wedge \neg Q(s)$$

Any Venn diagram that shades any region of $R \cap P \cap \neg Q$ makes the original sentence false.

(d) Some scholar is either not both pedantic and a quibbler, or is redundant.

$$\neg(\exists s \in S, \neg(P(s) \wedge Q(s)) \vee R(s)) \Leftrightarrow \forall s \in S, P(s) \wedge Q(s) \wedge \neg R(s)$$

Any Venn diagram that does not shade outside of $P \cap Q \cap \neg R$ makes the original sentence false.

(e) Whenever any scholar is a quibbler, there is a scholar who is redundant and pedantic.

$$\neg((\exists s \in S, Q(s)) \Rightarrow (\exists t \in S, R(t) \wedge P(t))) \Leftrightarrow (\exists s \in S, Q(s)) \wedge (\forall t \in S, \neg R(t) \vee \neg P(t))$$

Any Venn diagram that shades some region of Q and does not shade $R \cap P$ makes the original sentence false.

5. In the previous question, is (a) equivalent to the negation of (b)? Either provide a counter-example or show they are equivalent.

   Suppose there are some pedantic scholars who are neither redundant nor quibblers, and these are the only category of scholars there are. Then sentence (a) is true, since its consequent (being pedantic) is true for all scholars. Also, sentence (b) is true, since its consequent (being non-redundant) is true for some scholar. Since the universe described makes (a) and (b) true, (a) is not equivalent to the negation of (b).