

First, let's write down the forward pass. The variables are:

- `xs` the input sequence, encoded using one-hot encoding. Denote it by  $x_t$ .
- `hs` the hidden state (a vector), at each time step. Denote it by  $h_t = \tanh(W^{xh}x_t + W^{hh}h_{t-1})$
- `ys` the output layer. Denote it by  $y_t = W^{hy}h_t + b^y$
- `ps` the output of the softmax. Denote it by  $\hat{y}_t = \text{softmax}(y_t)$
- `loss` the cost/loss function.  $Cost = -\sum_t \log(\sum_k \hat{y}_t^k x_t^k)$

Now, let's go line by line and interpret those. We will often use e.g.  $\partial Cost_t / \partial h$  to denote the contribution from time-step  $t$  to the cost function, with  $C = \sum_t C_t$  for

$$C_t = \log\left(\sum_k \hat{y}_t^k x_t^k\right).$$

```
dy = np.copy(ps[t])
dy[targets[t]] -= 1 # backprop into y
```

This is just the derivative of the softmax for  $Cost_t$ :

$$\frac{\partial Cost_t}{\partial y} = \hat{y} - x$$

Note that  $x$  is one-hot encoded, so that it's mostly zeros, with only a single 1 at coordinate `targets[t]`. That's why we first set `dy` to `ps` (i.e., the  $\hat{y}$ ), and then subtract  $y$ .

```
dWhy += np.dot(dy, hs[t].T)
dby += dy
```

This corresponds to the  $t$ -th component of the derivatives wrt  $W^{hy}$  and  $b^y$ :

$$\begin{aligned} \partial Cost_t / \partial W^{hy} &= \frac{\partial Cost_t}{\partial y_t} \frac{\partial y_t}{\partial W^{hy}} = \frac{\partial Cost_t}{\partial y_t} h_t^T \\ \partial Cost_t / \partial W^{hy} &= \frac{\partial Cost_t}{\partial y_t} \frac{\partial y_t}{\partial b^y} = \frac{\partial Cost_t}{\partial y_t} 1 = \frac{\partial Cost_t}{\partial y_t} \end{aligned}$$

```
dh = np.dot(Why.T, dy) + dhnext
```

This is tricky. We want to account for the influence of  $h_t$  on both  $Cost_t$  and  $Cost_{(t+1):end}$ .

$$\frac{\partial Cost_{t:end}}{\partial h_t} = \frac{\partial Cost_t}{\partial h_t} + \frac{\partial Cost_{(t+1):end}}{\partial h_t} = \frac{\partial Cost_t}{\partial y} \frac{\partial y}{\partial h_t} + dhnext$$

```
dhraw = (1 - hs[t] * hs[t]) * dh
```

$$\frac{\partial Cost_{t:end}}{\partial hraw_t} = (1 - h_t^2) \frac{\partial Cost_{t:end}}{\partial h_t}$$

The following:

```
dbh += dhrw
dWxh += np.dot(dhrw, xs[t].T)
dWhh += np.dot(dhrw, hs[t-1].T)
```

are similar to what we already had. Note that

$$\frac{\partial Cost_{t:end}}{\partial h_t} = \frac{\partial Cost}{\partial h_t}$$

since  $h_t$  cannot influence components of the cost that come before it in time.

Finally, we compute `dhnnext`, which must be  $\frac{\partial Cost_{t:end}}{\partial h_{t-1}}$  in order for our earlier definition to work. Now

$$\frac{\partial Cost_{t:end}}{\partial h_{t-1}} = \frac{\partial Cost_{t:end}}{\partial hrw_t} \frac{\partial hrw_t}{\partial h_{t-1}}$$

This is exactly what the following line does.

```
dhnnext = np.dot(Whh.T, dhrw)
```

The following is self-explanatory:

```
for dparam in [dWxh, dWhh, dWhy, dbh, dby]:
```

```
    np.clip(dparam, -5, 5, out=dparam) # clip to mitigate exploding gradients
```

In the loop, we are adding up all the contributions to the gradients from all the time-steps  $t$ .