Image Warping

Salvador Dalí, "The Persistence of Memory"

CSC320: Introduction to Visual Computing

Michael Guerzhoy

Many slides from Derek Hoiem, Alyosha Efros, Steve Seitz

Morphing

Blend from one object to other with a series of local transformations

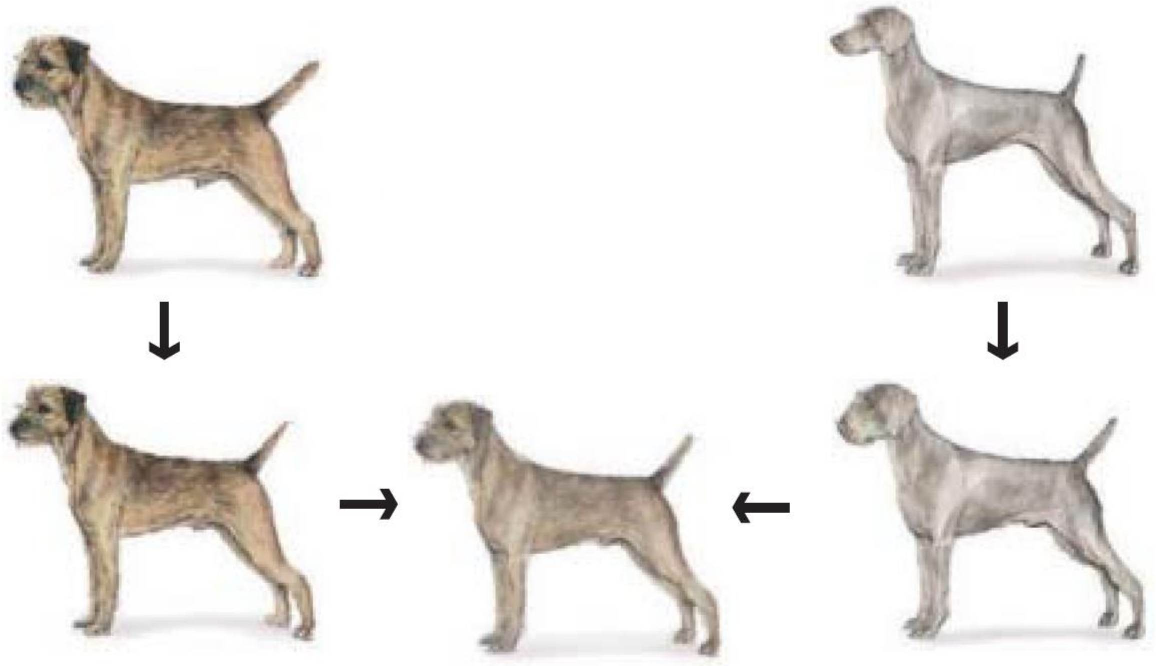


Image Transformations

image filtering: change ***range*** of image

$$g(x) = T(f(x))$$

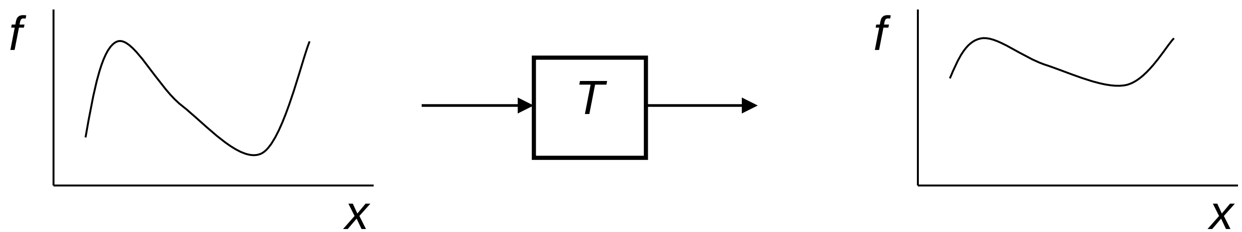


image warping: change ***domain*** of image

$$g(x) = f(T(x))$$

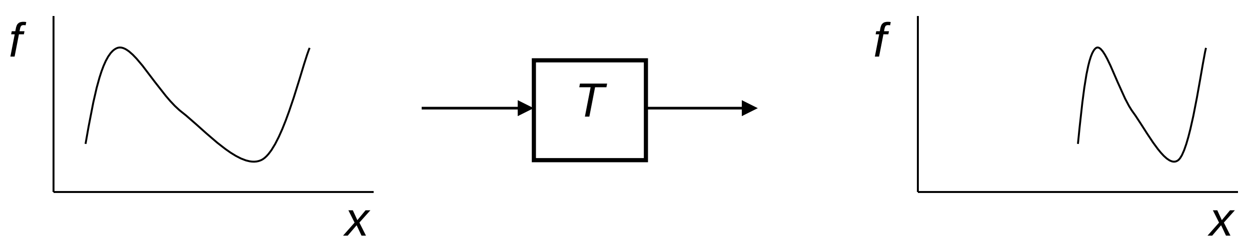


Image Transformations

image filtering: change ***range*** of image

$$g(x) = T(f(x))$$

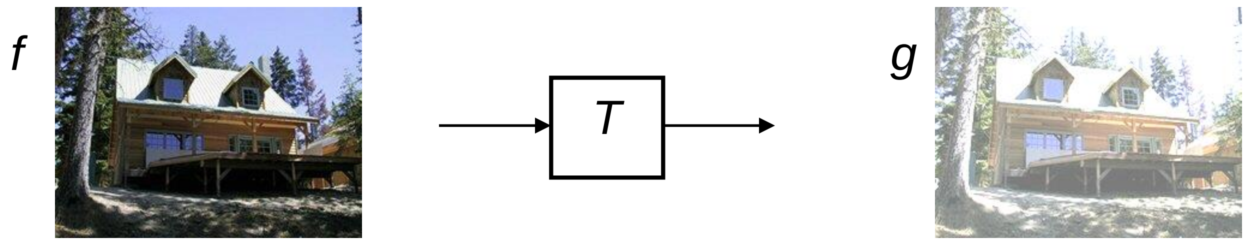
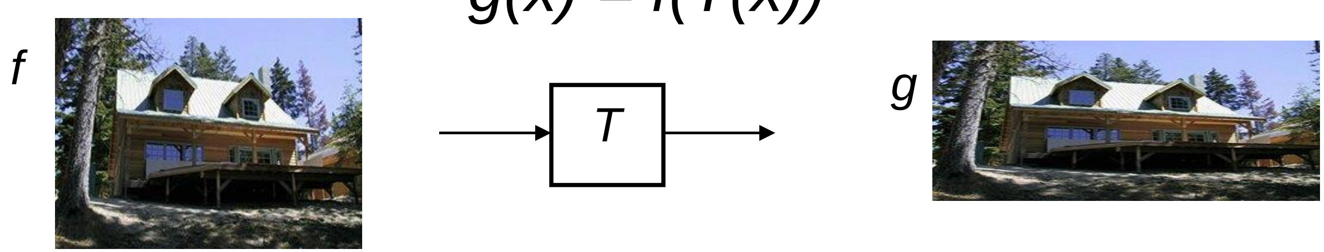
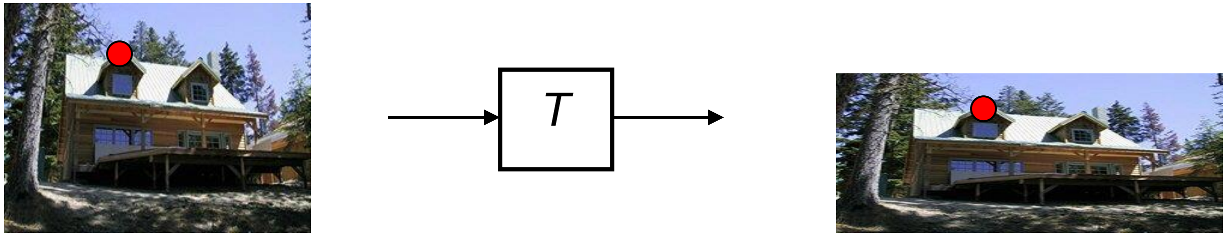


image warping: change ***domain*** of image

$$g(x) = f(T(x))$$



Parametric (global) warping



p = (x,y) **p'** = (x',y')

Transformation T is a coordinate-changing machine:

$$p' = T(p)$$

What does it mean that *T* is global?

- Is the same for any point p
- can be described by just a few numbers (parameters)

For linear transformations, we can represent T as a matrix

$$p' = \mathbf{M}p$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

Parametric (global) warping

Examples of parametric warps:

translation

rotation

aspect

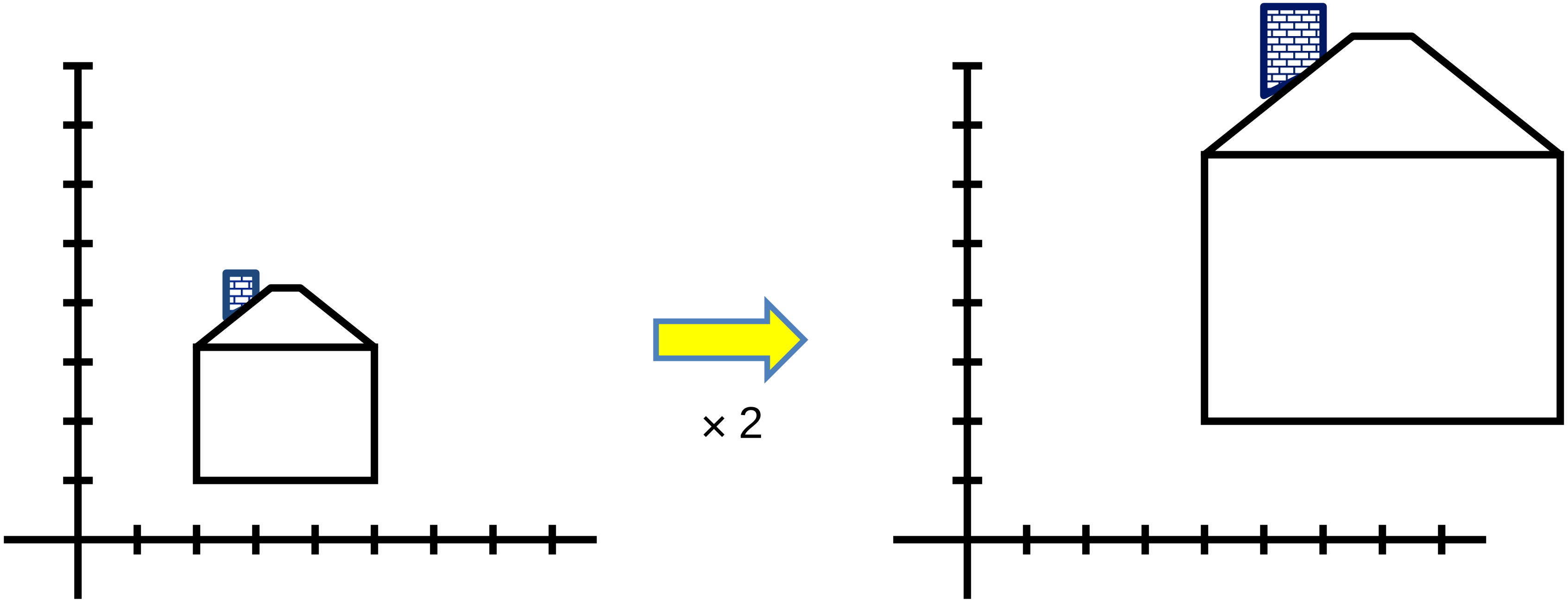
affine

perspective

cylindrical

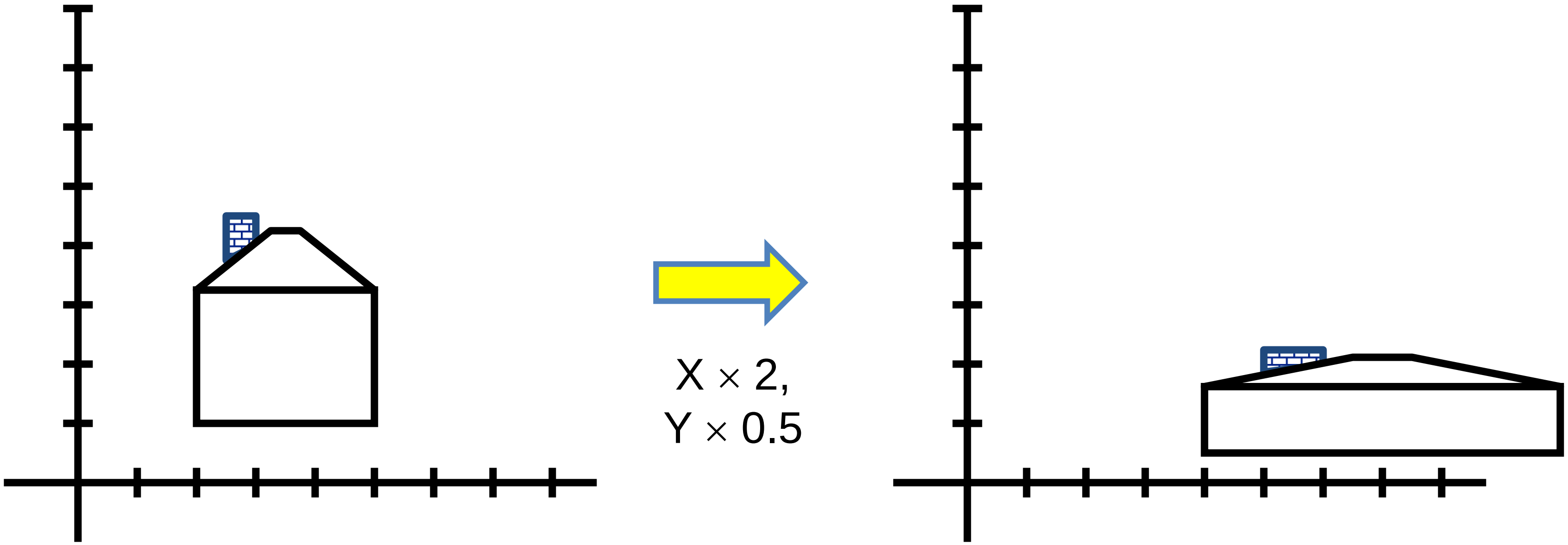
Scaling

- *Scaling* a coordinate means multiplying each of its components by a scalar
- *Uniform scaling* means this scalar is the same for all components:



Scaling

- *Non-uniform scaling*: different scalars per component:



Scaling

- Scaling operation:

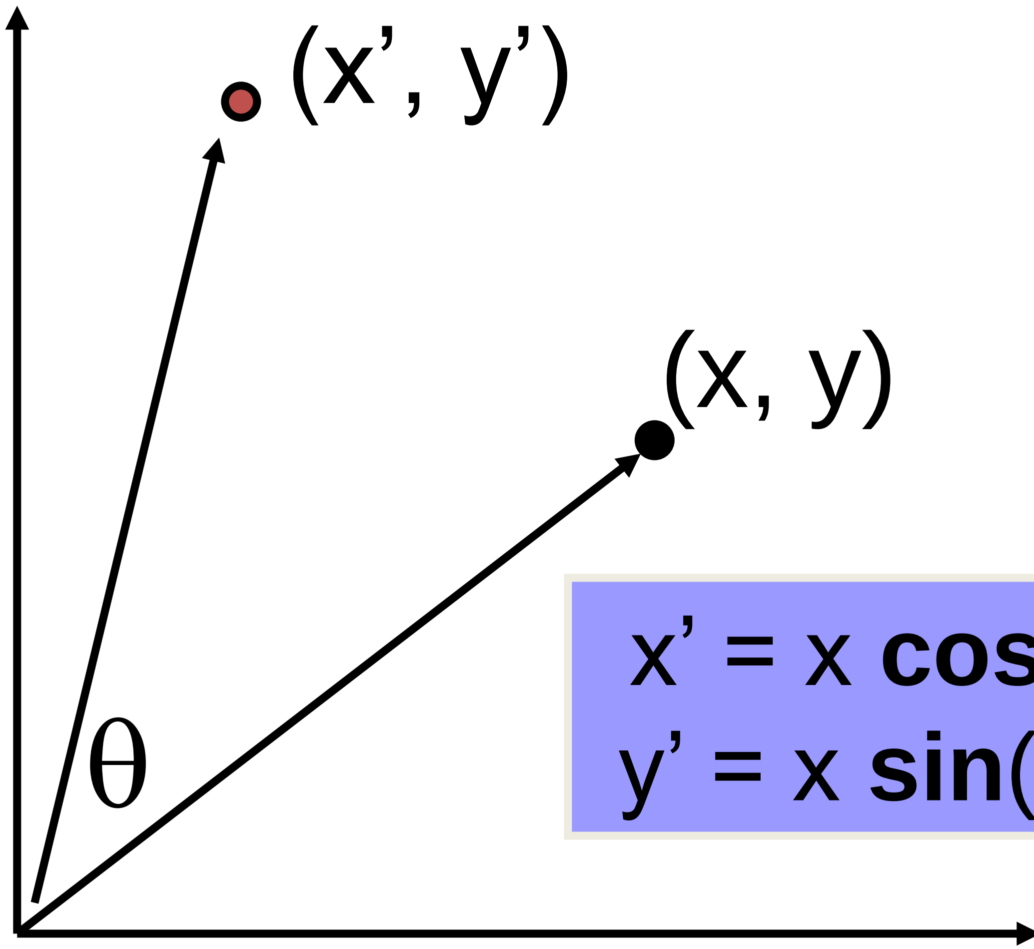
$$x' = ax$$
$$y' = by$$

- Or, in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\textit{scaling matrix S}} \begin{bmatrix} x \\ y \end{bmatrix}$$

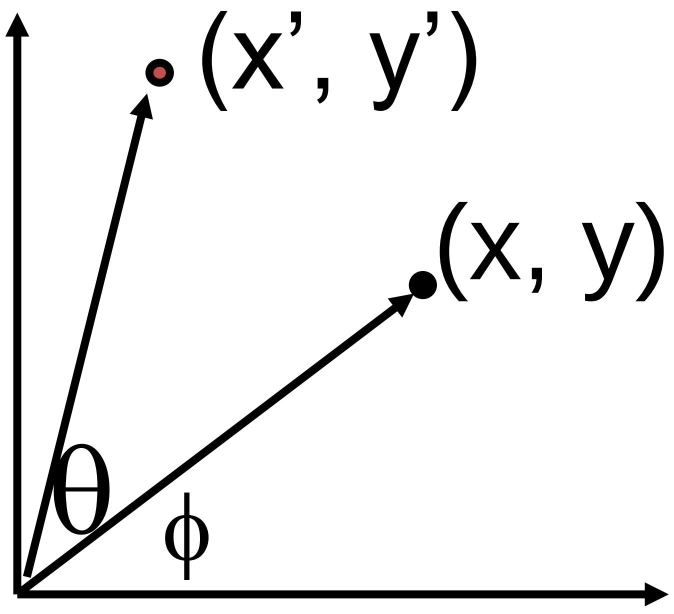
What is the transformation from (x', y') to (x, y)?

2-D Rotation



$$x' = x\,\mathbf{cos}(\theta) - y\,\mathbf{sin}(\theta)$$
$$y' = x\,\mathbf{sin}(\theta) + y\,\mathbf{cos}(\theta)$$

2-D Rotation



Polar coordinates…

$x = r \cos(\phi)$

$y = r \sin(\phi)$

$x' = r \cos(\phi + \theta)$

$y' = r \sin(\phi + \theta)$

Trig Identity…

$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$

$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$

Substitute…

$x' = x\, \mathbf{cos}(\theta) - y\, \mathbf{sin}(\theta)$

$y' = x\, \mathbf{sin}(\theta) + y\, \mathbf{cos}(\theta)$

2-D Rotation

This is easy to capture in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Even though sin(θ) and cos(θ) are nonlinear functions of θ,

- ***x' is a linear combination of x and y***
- ***y' is a linear combination of x and y***

What is the inverse transformation?

- Rotation by –θ
- For rotation matrices $\mathbf{R}^{-1} = \mathbf{R}^{T}$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$x' = x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$\boldsymbol{x}' = \boldsymbol{s}_x * \boldsymbol{x}$$
$$\boldsymbol{y}' = \boldsymbol{s}_y * \boldsymbol{y}$$

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_x & 0 \\ 0 & \boldsymbol{s}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$x' = \cos\Theta * x - \sin\Theta * y$$
$$y' = \sin\Theta * x + \cos\Theta * y$$

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$$

2D Shear?

$$x' = x + k_x * y$$
$$y' = k_y * x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & k_x \\ k_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Mirror about Y axis?

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$x' = -x$$
$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + t_x$$

$$y' = y + t_y$$

NO!

Only linear 2D transformations can be represented with a 2x2 matrix

All 2D Linear Transformations

- Linear transformations are combinations of …
  - Scale,
  - Rotation,
  - Shear, and
  - Mirror
- Properties of linear transformations:
  - Origin maps to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Closed under composition

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Homogeneous Coordinates

Q: How can we represent translation in matrix form?

$$x' = x + t_x$$
$$y' = y + t_y$$

Homogeneous Coordinates

Homogeneous coordinates

- represent coordinates in 2 dimensions with a 3-vector

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\textbf{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

2D Points → Homogeneous Coordinates

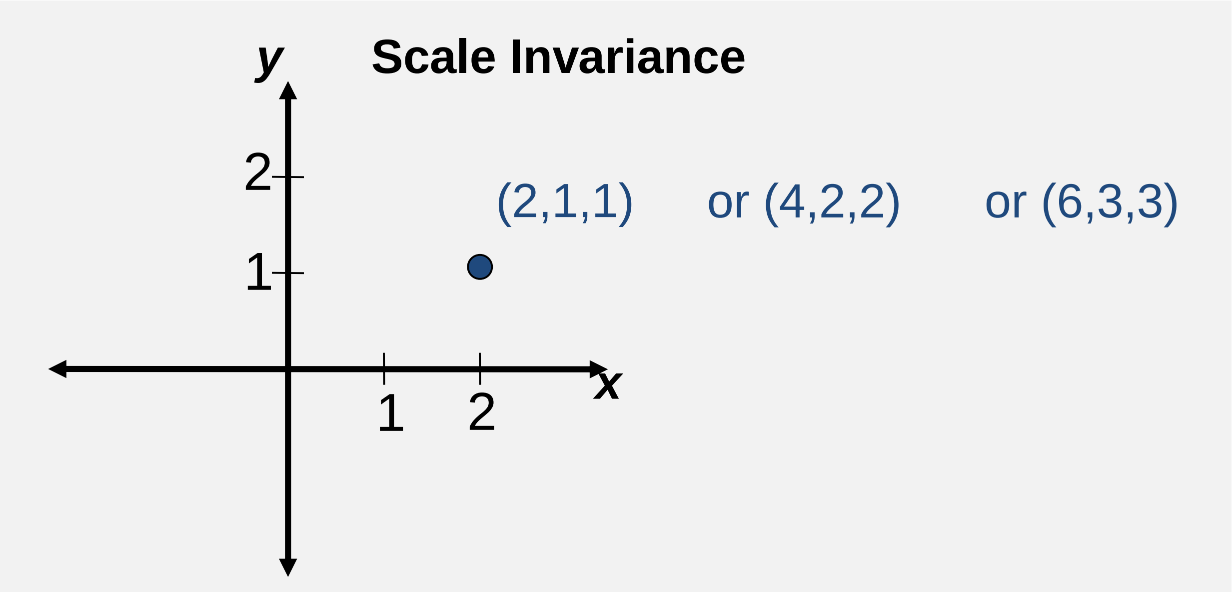
- Append 1 to every 2D point: (x y) → (x y 1)

Homogeneous coordinates → 2D Points

- Divide by third coordinate (x y w) → (x/w y/w)

Special properties

- Scale invariant: (x y w) = k * (x y w)
- (x, y, 0) represents a point at infinity
- (0, 0, 0) is not allowed



Homogeneous Coordinates

Q: How can we represent translation in matrix form?

$$x' = x + t_x$$

$$y' = y + t_y$$

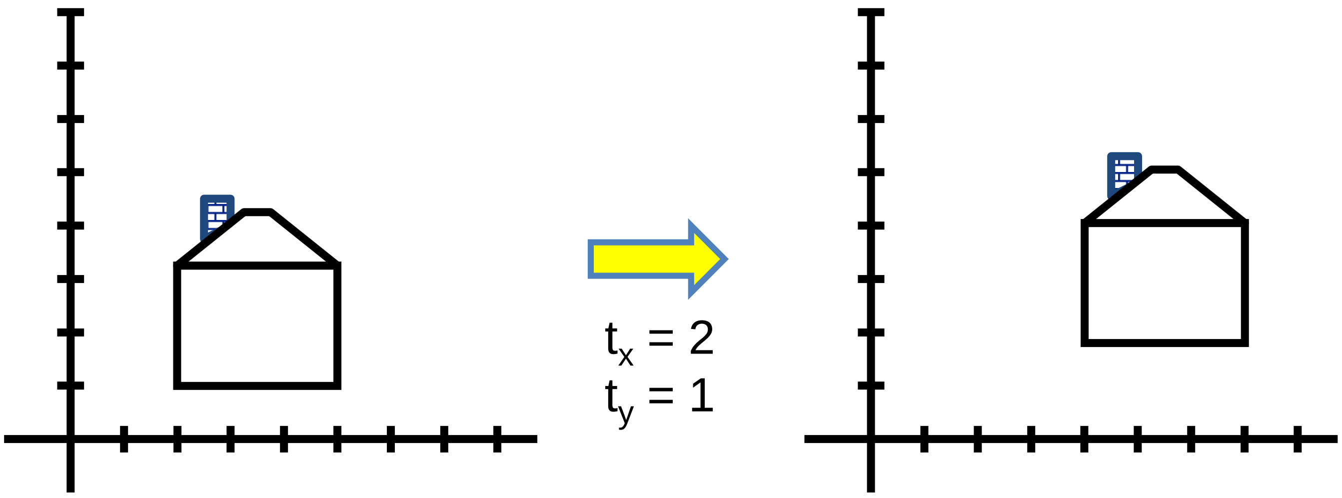
A: Using the rightmost column:

$$\mathbf{Translation} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Translation Example

Homogeneous Coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_x & 0 & 0 \\ 0 & \boldsymbol{s}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Matrix Composition

Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = $T(t_x,t_y)$ $R(\Theta)$ $S(s_x,s_y)$ **p**

Does the order of multiplication matter?

Affine Transformations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Affine transformations are combinations of

- Linear transformations, and
- Translations

Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines remain parallel
- Closed under composition

Projective Transformations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

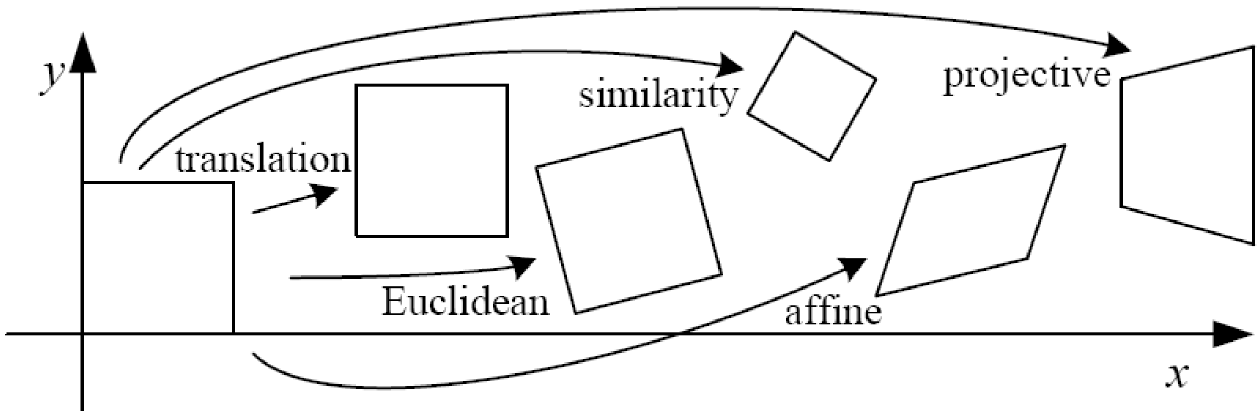
Projective transformations are combos of

- Affine transformations, and
- Projective warps

Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Closed under composition
- Models change of basis
- Projective matrix is defined up to a scale (8 DOF)

2D image transformations

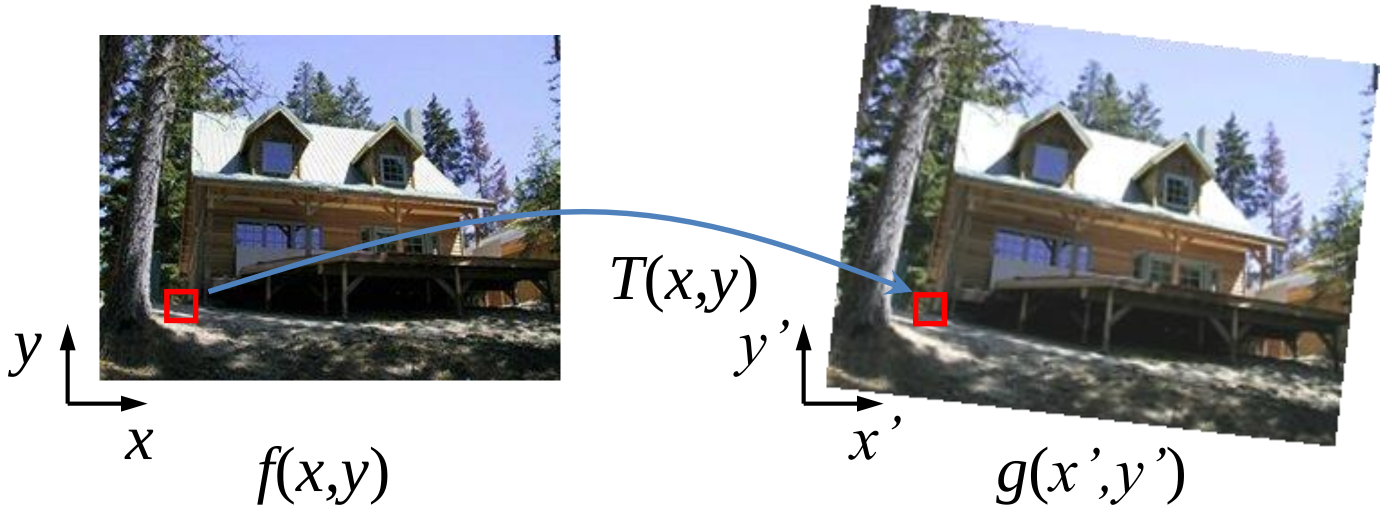


Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} \boldsymbol{I} & \vert & \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation $+ \cdots$	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} & \vert & \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths $+ \cdots$	
similarity	$\begin{bmatrix} s\boldsymbol{R} & \vert & \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles $+ \cdots$	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism $+ \cdots$	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

These transformations are a nested set of groups

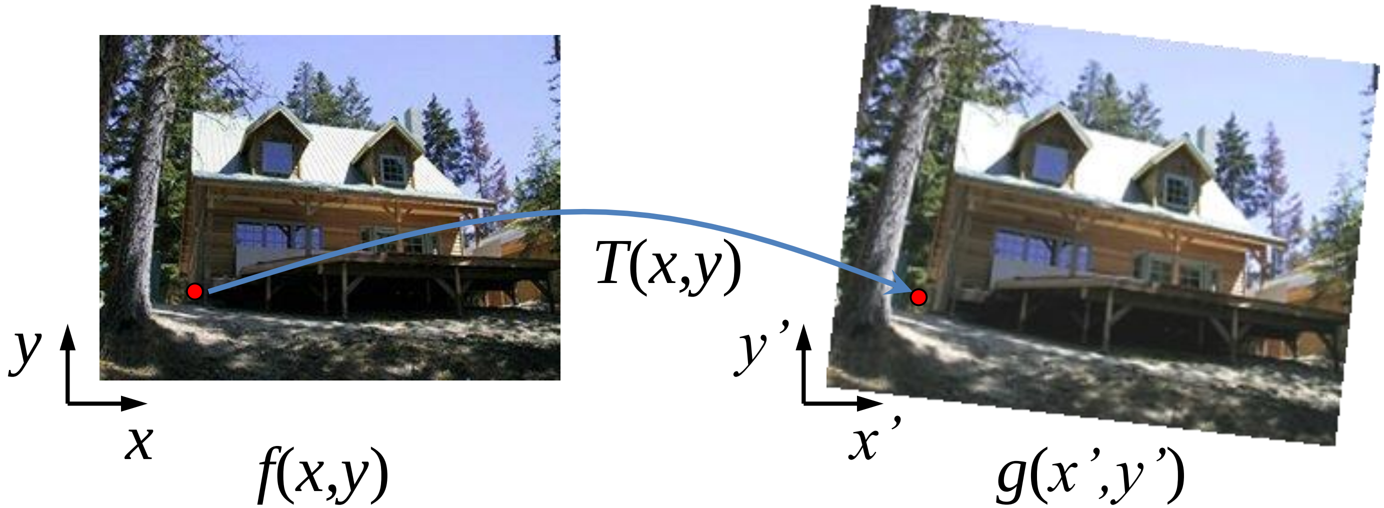
- Closed under composition and inverse is a member

Image warping



- Given a coordinate transform $(x',y') = T(x,y)$ and a source image $f(x,y)$, how do we compute a transformed image $g(x',y')$?

Forward warping

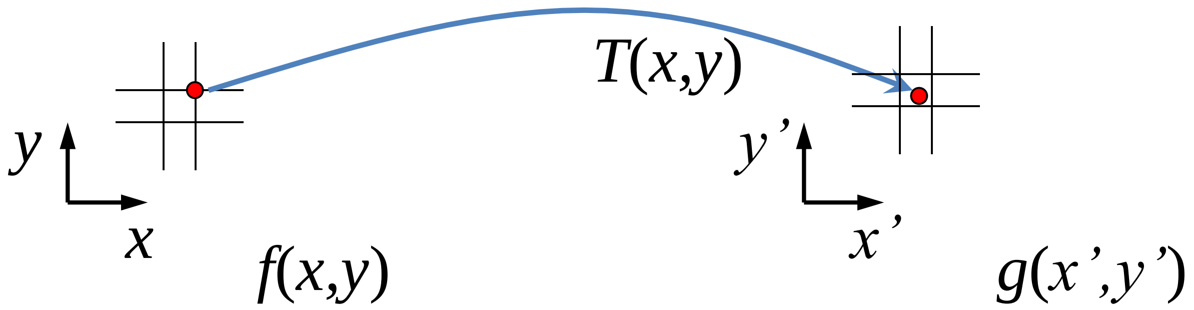


- Send each pixel $f(x,y)$ to its corresponding location
- $(x',y') = T(x,y)$ in the second image

Forward warping

$(x',y') = T(x,y)$ in the second image

What is the problem with this approach?



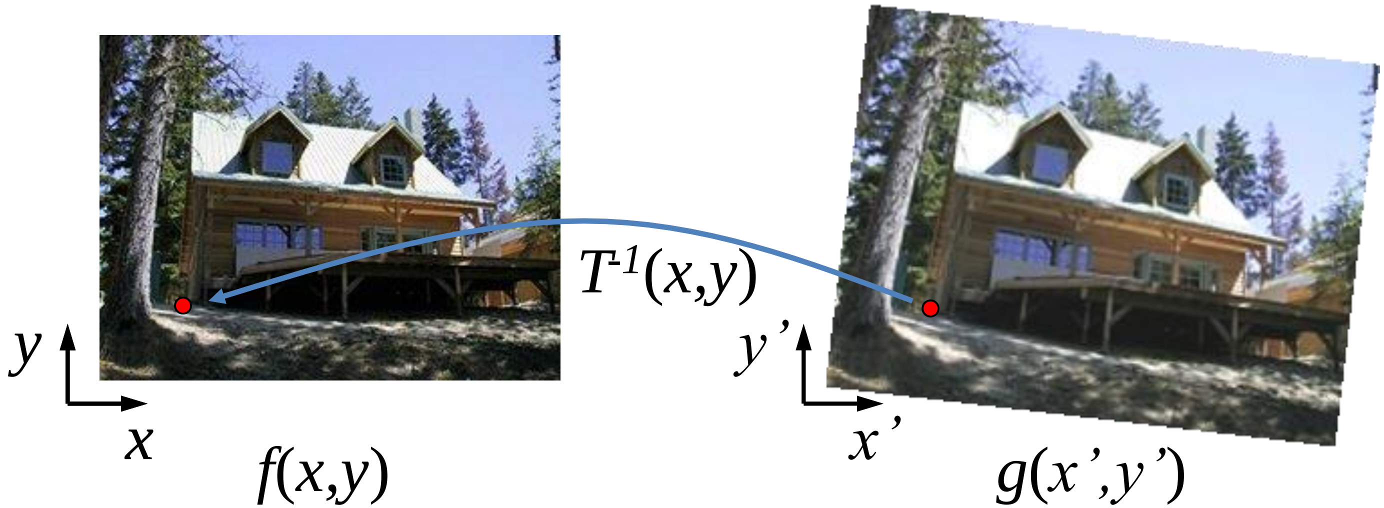
- Send each pixel $f(x,y)$ to its corresponding location

Q: what if pixel lands "between" two pixels?

A: distribute color among neighboring pixels (x',y')

- – Known as "splatting"

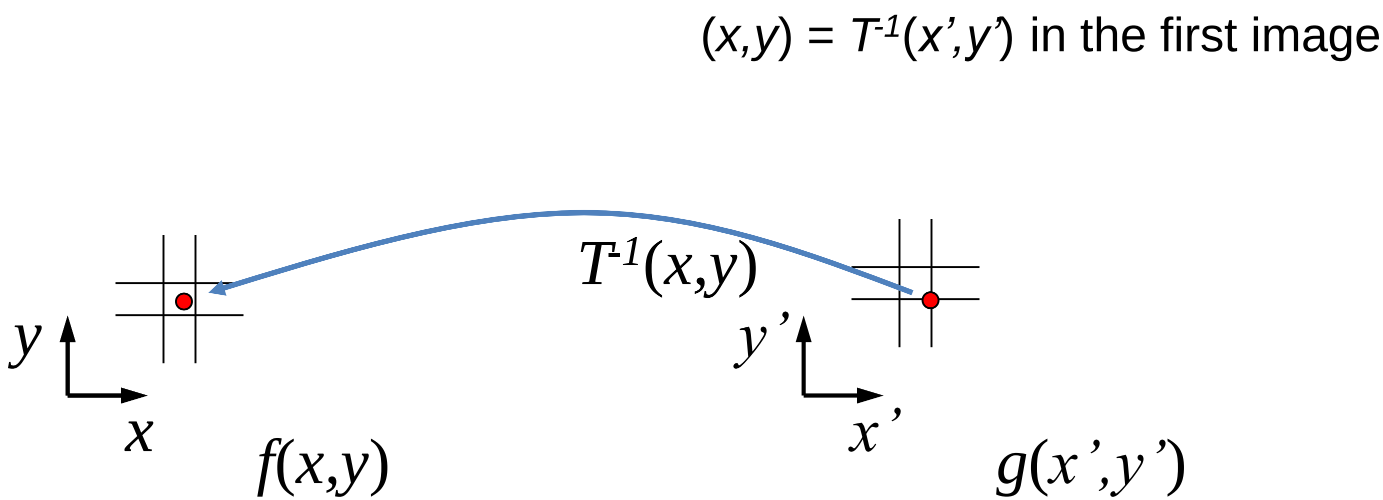
Inverse warping



- Get each pixel $g(x',y')$ from its corresponding location
- $(x,y) = T^{-1}(x',y')$ in the first image

Q: what if pixel comes from “between” two pixels?

Inverse warping



- Get each pixel $g(x',y')$ from its corresponding location

Q: what if pixel comes from "between" two pixels?

A: *Interpolate* color value from neighbors

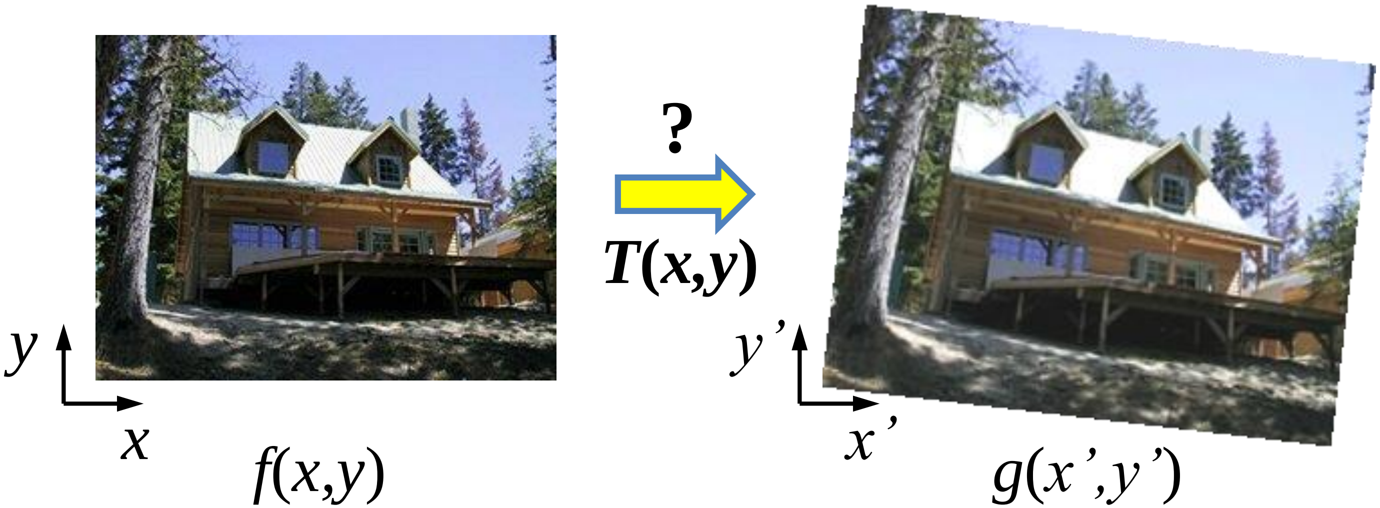
- nearest neighbor, bilinear, Gaussian, bicubic
- E.g. scipy.interpolate.interp2d

Forward vs. inverse warping

- Q: which is better?

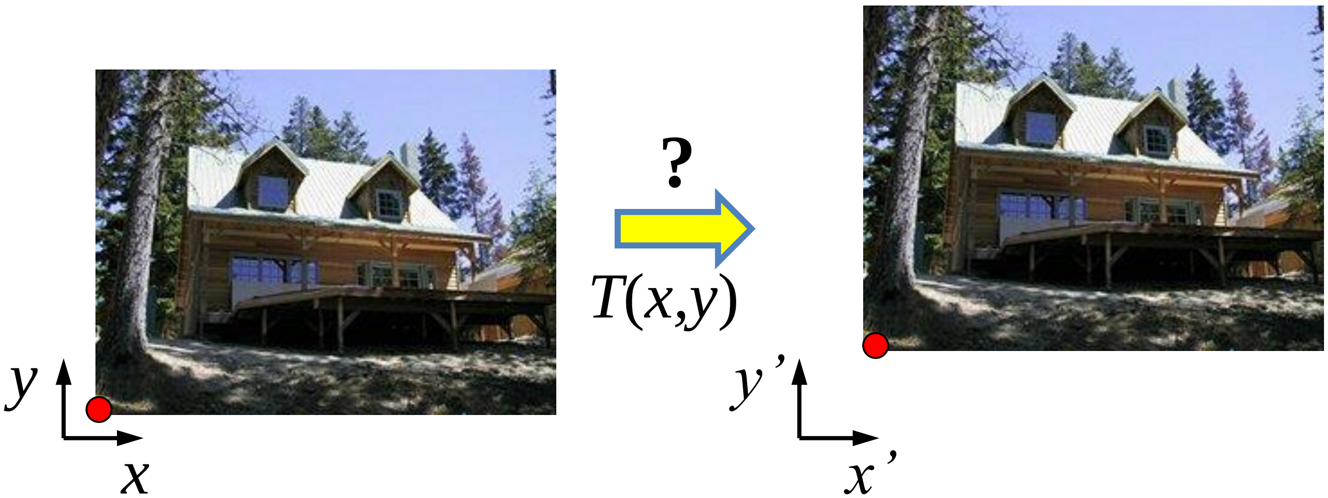
- A: Usually inverse—eliminates holes
  - however, it requires an invertible warp function

Recovering Transformations



- What if we know *f* and *g* and want to recover the transform T?
  - willing to let user provide correspondences
    - How many do we need?

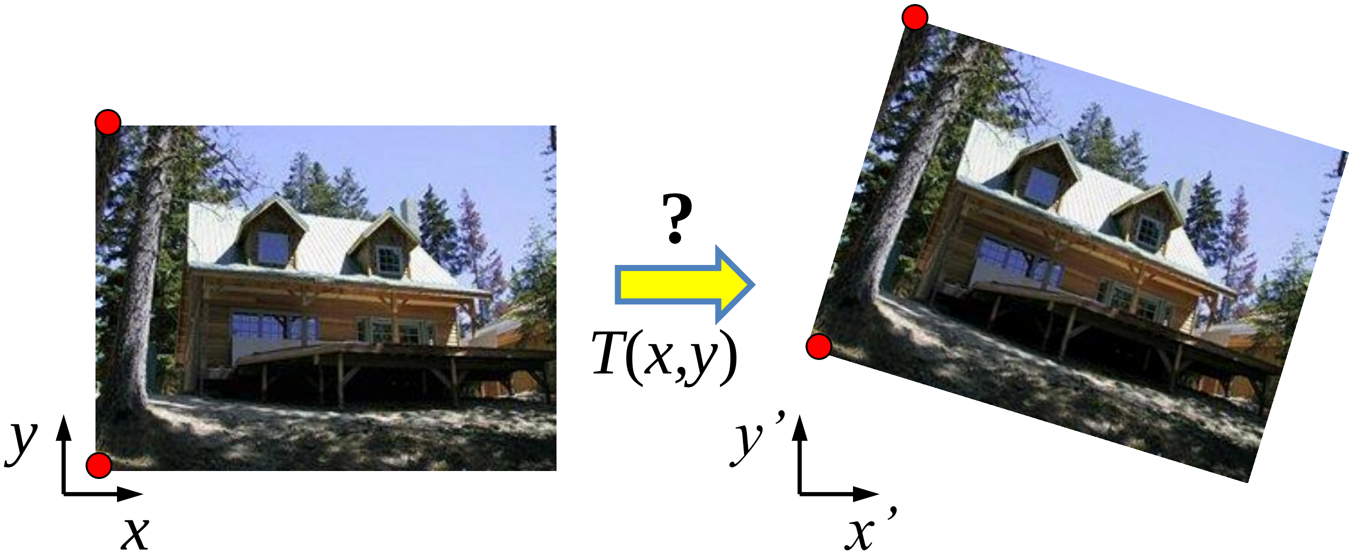
Translation: # correspondences?



- How many Degrees of Freedom?
- How many correspondences needed for translation?
- What is the transformation matrix?

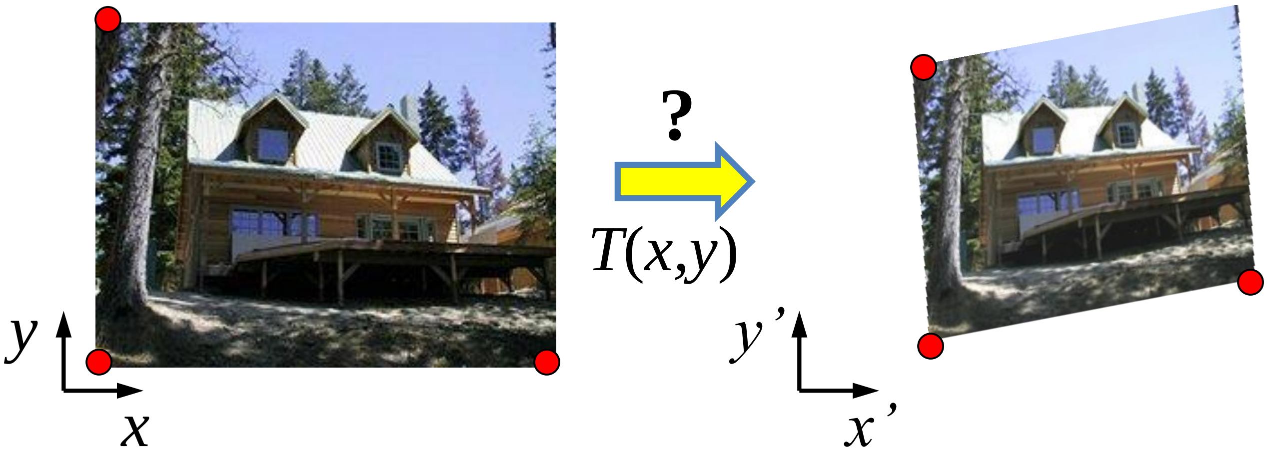
$$\mathbf{M} = \begin{bmatrix} 1 & 0 & p'_x - p_x \\ 0 & 1 & p'_y - p_y \\ 0 & 0 & 1 \end{bmatrix}$$

Euclidian: # correspondences?



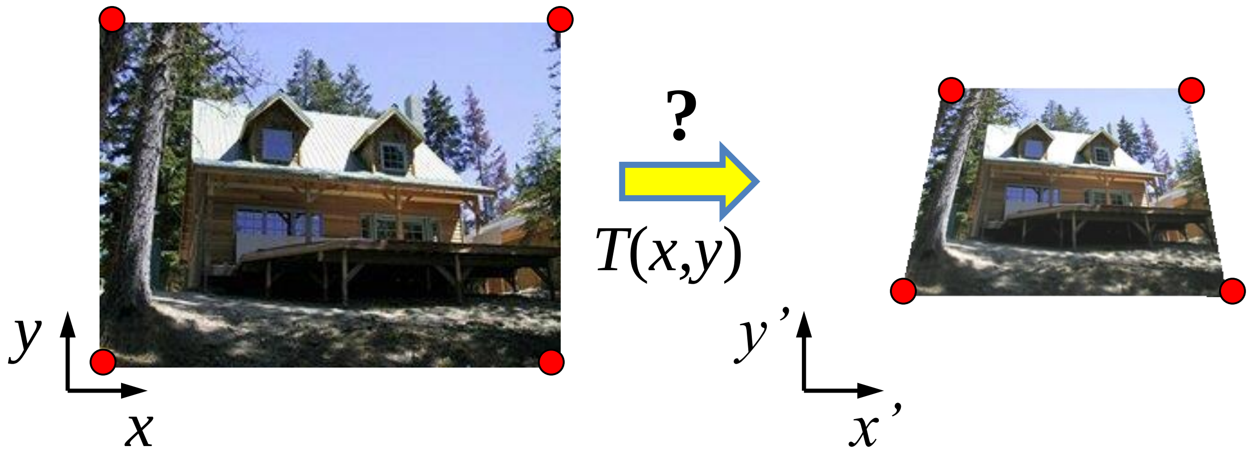
- How many DOF?
- How many correspondences needed for translation+rotation?

Affine: # correspondences?



- How many DOF?
- How many correspondences needed for affine?

Projective: # correspondences?



- How many DOF?
- How many correspondences needed for projective?