Question 1. [15 MARKS]

An "interpretation" for a logical statement consists of a domain D (any non-empty set of elements) and a meaning for each predicate symbol. For example, $D = \{1, 2\}$ and $P(x)$: "$x > 0$" is an interpretation for the statement $\forall x \in D, P(x)$ (in this case, one that happens to make the statement true). For each statement below, provide one interpretation under which the statement is true and another interpretation under which the statement is false— if either case is not possible, explain why clearly and concisely. You may reuse examples if you wish.

Part (a) [5 MARKS]
$\forall x \in D, \exists y \in D, P(x, y)$

Solution:
True: $D = \{1\}$, $P(x, y) = $ True
False: $D = \{1\}$, $P(x, y) = $ False

Part (b) [5 MARKS]
$[\exists x \in D, [P(x) \Rightarrow [\forall y \in D, Q(x, y)]]] \land [\exists x \in D, P(x)] \land [\exists x \in D, \neg P(x)]$

Solution:
True: $D = \{1, 2\}$, $P(1) = $ True, $P(2) = $ False, $Q(1, 1) = Q(1, 2) = Q(2, 1) = Q(2, 2) = $ True
False: $D = \{1, 2\}$, $P(1) = $ True, $P(2) = $ False, $Q(1, 1) = Q(1, 2) = Q(2, 1) = Q(2, 2) = $ False

Part (c) [5 marks]

$[\forall x \in D, P(x) \Rightarrow Q(x)] \wedge [\forall x \in D, P(x)] \wedge [\forall x \in D, \neg Q(x)]$

Solution:

True: this is impossible. Since D is not empty, there exists an $x \in D$. For the interpretation to be true, all of $P(x) \Rightarrow Q(x)$, $P(x)$, and $\neg Q(x)$ must be true, which means that both $P(x) \Rightarrow Q(x)$ and $P(x) \wedge \neg Q(x)$ must be true, which means, by the implication equivalence, that both $P(x) \Rightarrow Q(x)$ and $\neg[P(x) \Rightarrow Q(x)]$ must be true, but that's impossible since $R \wedge \neg R$ is False for any R.

False: $D = \{1\}$, $P(1) =$ True, $Q(1) =$ True.

Question 2. [5 marks]

Note: this is a challenging question. Don't get stuck on it!

Express the following fact using symbolic notation (i.e., using notation defined in this course): 1729 is the smallest natural number that is expressible as the sum of two cubes of positive natural numbers in two different ways. (Just for fun: those two ways are $1729 = 1^3 + 12^3$ and $1729 = 9^3 + 10^3$. The number 1729 is known as the Hardy-Ramanujan number.)

Solution:

$$H(n) : [\exists n_1 \in \mathbb{N}, \exists n_2 \in \mathbb{N}, \exists n_3 \in \mathbb{N}, \exists n_4 \in \mathbb{N}, [n = n_1^3 + n_2^3] \wedge [n = n_3^3 + n_4^4] \wedge [n_1 \neq n_3] \wedge [n_1 \neq n_4]]$$

$$H(1729) \wedge [\forall n \in \mathbb{N}, H(n) \Rightarrow [n \geqslant 1729]]$$

Question 3. [15 marks]

Part (a) [5 marks]

Consider the following argument that $\sqrt{2}/2$ is irrational. A number x is rational if there are integers p and q such that $x = p/q$. The number 2 is an integer but $\sqrt{2}$ is not an integer, so $\sqrt{2}/2$ can't be rational, so it is irrational.

This argument is incorrect (though the conclusion that $\sqrt{2}/2 \notin \mathbb{Q}$ is correct). Briefly explain the main error in this argument.

Solution:
It is not true that if $x = p/q$ and p is not an integer, than x is not rational. For example, $x_1 = (2\sqrt{2})/\sqrt{2}$ is rational. So is $x_2 = .5/.25 = 2 = 2/1$.

Part (b) [5 marks]

Consider the following argument that $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y < x$. Assume x is natural. Then we need to find a natural number y that is smaller than x in order for the statement to be true. Set $y = x - 1$. Now y is always smaller than x, so the y with the required properties exists, so the statement is true.

This argument is incorrect. Briefly explain the main error in this argument.

Solution:
The number $y = x - 1$ is not always natural, and y must be natural for the statement to be true. In particular, if $x = 0$, then $y = x - 1 = -1 \notin \mathbb{N}$, and the statement is false.

Part (c)   [5 marks]

Consider the following argument that, for any set U and predicates P and Q,

$$[[\exists x \in U, P(x)] \wedge [\exists x \in U, Q(x)]] \Rightarrow [\exists x \in U, P(x) \wedge Q(x)]$$

.

      Assume U is a set, P and Q are predicates
           Assume $[\exists x \in U, P(x)] \wedge [\exists x \in U, Q(x)]$
                Let $x \in U$ be such that P(x)
                Then P(x)
                Let $x \in U$ be such that $Q(x)$
                Then Q(x)
                Then $P(x) \wedge Q(x)$
           Then $\exists x \in U, P(x) \wedge Q(x)$
      Then $[[\exists x \in U, P(x)] \wedge [\exists x \in U, Q(x)]] \Rightarrow [\exists x \in U, P(x) \wedge Q(x)]$

This argument is incorrect. Briefly explain the main error in this argument.

Solution:

The variable x is being used to store (potentially) two different values (one x such that $P(x)$ is true, and one x such that $Q(x)$ is true). So the proof does not make sense.

Question 4. [10 marks]

Recall that a real number x is rational (i.e., $x \in \mathbb{Q}$) if $\exists p \in \mathbb{Z}, \exists q \in \mathbb{Z}^*, x = p/q$.

Express in symbolic form, and then prove or disprove the following claim: the product of any two rational numbers is a rational number. Give a detailed structured proof, justifying every step. Note: you may use the fact that the integers are closed under addition and multiplication. You may not assume, without proof, that the same is true about rational numbers.

Solution:

In symbolic form:
$$\forall r_1 \in \mathbb{Q}, \forall r_2 \in \mathbb{Q}, r_1 r_2 \in \mathbb{Q}$$

Assume $r_1 \in \mathbb{Q}$ and $r_2 \in \mathbb{Q}$
$\qquad$ Then $\exists p_1' \in \mathbb{N}, q_1' \in \mathbb{N}, r1 = p_1/q_1$ $\quad$ # Definition of $\mathbb{Q}$
$\qquad$ Also $\exists p_2' \in \mathbb{N}, q_2' \in \mathbb{N}, r2 = p_2/q_2$ $\quad$ # Definition of $\mathbb{Q}$
$\qquad$ Let $p_1 \in \mathbb{N}, q_1 \in \mathbb{N}$ be s.t. $r_1 = p_1/q_1$ $\quad$ # instantiation
$\qquad$ Let $p_2 \in \mathbb{N}, q_2 \in \mathbb{N}$ be s.t. $r_2 = p_2/q_2$ $\quad$ # instantiation
$\qquad$ Now $p_1 p_2 \in \mathbb{N}$ $\quad$ # integers closed under multiplication
$\qquad$ Now $q_1 q_2 \in \mathbb{N}$ $\quad$ # integers closed under multiplication
$\qquad$ Also $r_1 r_2 = \frac{p_1 p_2}{q_1 q_2}$ $\quad$ # fraction multiplication
$\qquad$ Then $\exists p \in \mathbb{N}, \exists q \in \mathbb{N}, r_1 r_2 = p/q$ $\quad$ # $p_1 p_2$ and $q_1 q_2$ are such p and q
$\qquad$ Then $r_1 r_2 \in \mathbb{Q}$ $\quad$ # definition of $\mathbb{Q}$
Then $\forall r_1 \in \mathbb{Q}, \forall r_2 \in \mathbb{Q}, r_1 r_2 \in \mathbb{Q}$ $\quad$ # introduce universal
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ∎

Question 5. [10 marks]

Recall that a real number x is irrational if it is not rational. Express in symbolic form, and then **prove** the following claim: the product of a nonzero rational number and an irrational number is an irrational number. Give a detailed structured proof, justifying every step.

Solution:

In symbolic form:
$$\forall r \in [\mathbb{Q} - \{0\}], \forall x \in \mathbb{R}, [x \notin \mathbb{Q} \Rightarrow rx \notin \mathbb{Q}]$$

Proof:

     Assume $r \in [\mathbb{Q} - \{0\}]$

         Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}, r = p'/q' \wedge p' \neq 0$ # definition of $\mathbb{Q}$, $r \neq 0$

         Let $p \in \mathbb{Z}, q \in \mathbb{Z}$ be s.t. $x = p/q$ # instantiation

         Assume $x \in \mathbb{R}$

             Assume $rx \in \mathbb{Q}$

                Then $xp/q \in \mathbb{Q}$ # substitution

                Then $\exists p' \in \mathbb{Z}, \exists q' \in mathbbZ$ s.t. $p'/q' = xp/q$ # definition of $\mathbb{Q}$

                Let $p_0 \in \mathbb{Z}, q_0 \in \mathbb{Z}$ be s.t. $xp/q = p_0/q_0$ # instantiation

                Then $xp/q = p_0/q_0$ # substitution

                Then $x = qp_0/pq_0$ # algebra, $p \neq 0$ so we can divide

                $qp_0 \in \mathbb{Z}, pq_0 \in \mathbb{Z}$ # integers closed under multiplication

                Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}$ s.t. $x = p'/q'$ # qp_0 and pq_0 are such p' and q'

                Then $x \in \mathbb{Q}$ # definition of $\mathbb{Q}$

                Then $rx \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$ # introduce implication

     Then $\forall r \in [\mathbb{Q} - \{0\}], \forall x \in \mathbb{R}, rx \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$ # introduce universal, twice

     Then $\forall r \in [\mathbb{Q} - \{0\}], \forall x \in \mathbb{R}, x \notin \mathbb{Q} \Rightarrow rx \notin \mathbb{Q}$ # contrapositive

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Question 6. [15 MARKS]

Express in symbolic form, and **disprove** the following claim: the sum of any two irrational numbers is an irrational number. Give a detailed structured proof, justifying every step. You may use, without proof, the fact that $\sqrt{2} \notin \mathbb{Q}$. You have to prove any other property of irrational numbers that you wish to use.

Solution:

In symbolic form:

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [[[x \notin \mathbb{Q}] \wedge [y \notin \mathbb{Q}]] \Rightarrow [x + y \notin \mathbb{Q}]]$$

The statement is **false**.

We first prove that $-\sqrt{2}$ is irrational:

$-\sqrt{2} = -1 \times \sqrt{2} \in \mathbb{R}$ # the reals are closed under multiplication, $\sqrt{2} \in \mathbb{R}, -1 \in \mathbb{R}$

Assume $-\sqrt{2} \in \mathbb{Q}$

     Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}, -\sqrt{2} = p'/q'$ # definition of $\mathbb{Q}$

     Let $p \in \mathbb{Z}, q \in \mathbb{Z}$ be s.t. $-\sqrt{2} = p/q$ # instantiation

     Then $\sqrt{2} = -p/q$ # algebra

     Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}, \sqrt{2} = p'/q'$ # $-p$ and q are such p' and q'

     Then $\sqrt{2} \in \mathbb{Q}$ # definition of $\mathbb{Q}$

     Contradiction!   # $\sqrt{2} \notin \mathbb{Q}$

Then $-\sqrt{2} \notin \mathbb{Q}$ # assuming otherwise leads to a contradiction   ∎

We can now prove the main result:

$\sqrt{2} \in \mathbb{R}, -\sqrt{2} \in \mathbb{R}, \in \mathbb{R}$ # properties of reals

$[\sqrt{2} \notin \mathbb{Q}] \wedge [-\sqrt{2} \notin \mathbb{Q}]$ # proven or assumed above

$\sqrt{2} + (-\sqrt{2}) = 0$ # definition of negation

$0 \in \mathbb{Q}$ # $0 = 0/1$ (could do it the long way...

Then $\neg[\sqrt{2} + (-\sqrt{2}) \notin \mathbb{Q}]$ # substitution

Then $[[\sqrt{2} \notin \mathbb{Q}] \wedge [-\sqrt{2} \notin \mathbb{Q}]] \wedge [\neg[\sqrt{2} + (-\sqrt{2}) \notin \mathbb{Q}]]$ # conjunction

Then $\neg[[[\sqrt{2} \notin \mathbb{Q}] \wedge [-\sqrt{2} \notin \mathbb{Q}]] \Rightarrow [\sqrt{2} + (-\sqrt{2}) \notin \mathbb{Q}]]$ # implication negation

Then $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \neg[[[x \notin \mathbb{Q}] \wedge [y \notin \mathbb{Q}]] \Rightarrow [x + y \notin \mathbb{Q}]]$ # $\sqrt{2}$ and $-\sqrt{2}$ are such x and y

Then $\neg[\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [[[x \notin \mathbb{Q}] \wedge [y \notin \mathbb{Q}]] \Rightarrow [x + y \notin \mathbb{Q}]]]$ # quantifier negation twice   ∎

Question 7. [20 MARKS]

Prove or disprove the following claim:

$$\forall x \in \mathbb{Z}, [(\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)]$$

Give a detailed structured proof, justifying every step.
Solution:

The statement is **true**.

Proof:

    Assume $x \in \mathbb{Z}$

        Assume $\exists y \in \mathbb{Z}, x = 3y + 1$

            Let $y' \in \mathbb{Z}$ be s.t. $x = 3y' + 1$ # instantiation

            Then $x^2 = (3y' + 1)^2 = 9y'^2 + 6y' + 1 = 3(3y'^2 + 2y') + 1$ # algebra

            $3y'^2 + 2y' \in \mathbb{Z}$ # the integers are closed under multiplication and addition

            Then $\exists y \in \mathbb{Z}, x^2 = 3y + 1$ # $3y'^2 + 2y' \in \mathbb{Z}$ is such a y

        Then $[\exists y \in \mathbb{Z}, x = 3y + 1] \Rightarrow [\exists y \in \mathbb{Z}, x^2 = 3y + 1]$ # assuming the antecedent leads to the consequent

    Then $\forall x \in \mathbb{Z}, [\exists y \in \mathbb{Z}, x = 3y + 1] \Rightarrow [\exists y \in \mathbb{Z}, x^2 = 3y + 1]$ # introduce universal

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Total Marks = 90