Duration: **3 hours (6:10pm – 9:10pm)**
Aids Allowed: **Formula sheet supplied with the test**

Student Number: |_|_|_|_|_|_|_|_|_|_|

Family Name(s): ______________________________

Given Name(s): ______________________________

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*Do **not** turn this page until you have received the signal to start.*
In the meantime, please fill out the identification section above.

MARKING GUIDE

1: ______/15

2: ______/ 5

3: ______/15

4: ______/10

5: ______/10

6: ______/15

7: ______/20

TOTAL: ______/90

This term test consists of 7 questions on 18 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, and **write your name on the back of the last page.***

You will receive 10% of the marks for answering "I don't know" to any questions or sub-question.

Good Luck

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Label any parts that you want marked with the appropriate question and part number.

Question 1. [15 MARKS]

An "interpretation" for a logical statement consists of a domain D (any non-empty set of elements) and a meaning for each predicate symbol. For example, $D = \{1, 2\}$ and $P(x)$: "$x > 0$" is an interpretation for the statement $\forall x \in D, P(x)$ (in this case, one that happens to make the statement true). For each statement below, provide one interpretation under which the statement is true and another interpretation under which the statement is false — if either case is not possible, explain why clearly and concisely. You may reuse examples if you wish.

Part (a) [5 MARKS]

$\forall x \in D, \exists y \in D, P(x, y)$

Part (b) [5 MARKS]

$[\exists x \in D, [P(x) \Rightarrow [\forall y \in D, Q(x, y)]]] \wedge [\exists x \in D, P(x)] \wedge [\exists x \in D, \neg P(x)]$

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Part (c) [5 MARKS]

$[\forall x \in D, P(x) \Rightarrow Q(x)] \wedge [\forall x \in D, P(x)] \wedge [\forall x \in D, \neg Q(x)]$

Question 2. [5 MARKS]

Note: this is a challenging question. Don't get stuck on it!

Express the following fact using symbolic notation (i.e., using notation defined in this course): 1729 is the smallest natural number that is expressible as the sum of two cubes of positive natural numbers in two different ways. (Just for fun: those two ways are $1729 = 1^3 + 12^3$ and $1729 = 9^3 + 10^3$. The number 1729 is known as the Hardy-Ramanujan number.)

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Label any parts that you want marked with the appropriate question and part number.

Question 3. [15 MARKS]

Part (a) [5 MARKS]

Consider the following argument that $\sqrt{2}/2$ is irrational. A number x is rational if there are integers p and q such that $x = p/q$. The number 2 is an integer but $\sqrt{2}$ is not an integer, so $\sqrt{2}/2$ can't be rational, so it is irrational.
This argument is incorrect (though the conclusion that $\sqrt{2}/2 \notin \mathbb{Q}$ is correct). Briefly explain the main error in this argument.

Part (b) [5 MARKS]

Consider the following argument that $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y < x$. Assume x is natural. Then we need to find a natural number y that is smaller than x in order for the statement to be true. Set $y = x - 1$. Now y is always smaller than x, so the y with the required properties exists, so the statement is true.
This argument is incorrect. Briefly explain the main error in this argument.

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
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Part (c) [5 MARKS]
Consider the following argument that, for any set U and predicates P and Q,

$$[[\exists x \in U, P(x)] \wedge [\exists x \in U, Q(x)]] \Rightarrow [\exists x \in U, P(x) \wedge Q(x)]$$

.

Assume U is a set, P and Q are predicates
  Assume $[\exists x \in U, P(x)] \wedge [\exists x \in U, Q(x)]$
    Let $x \in U$ be such that P(x)
    Then P(x)
    Let $x \in U$ be such that $Q(x)$
    Then Q(x)
    Then $P(x) \wedge Q(x)$
  Then $\exists x \in U, P(x) \wedge Q(x)$
Then $[[\exists x \in U, P(x)] \wedge [\exists x \in U, Q(x)]] \Rightarrow [\exists x \in U, P(x) \wedge Q(x)]$

This argument is incorrect. Briefly explain the main error in this argument.

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Label any parts that you want marked with the appropriate question and part number.

Question 4. [10 MARKS]

Recall that a real number x is rational (i.e., $x \in \mathbb{Q}$) if $\exists p \in \mathbb{Z}, \exists q \in \mathbb{Z}^*, x = p/q$.

Express in symbolic form, and then prove or disprove the following claim: the product of any two rational numbers is a rational number. Give a detailed structured proof, justifying every step. Note: you may use the fact that the integers are closed under addition and multiplication. You may not assume, without proof, that the same is true about rational numbers.

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Label any parts that you want marked with the appropriate question and part number.

Question 5. [10 MARKS]

Recall that a real number x is irrational if it is not rational. Express in symbolic form, and then **prove** the following claim: the product of a nonzero rational number and an irrational number is an irrational number. Give a detailed structured proof, justifying every step.

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Label any parts that you want marked with the appropriate question and part number.

Question 6. [15 MARKS]

Express in symbolic form, and **disprove** the following claim: the sum of any two irrational numbers is an irrational number. Give a detailed structured proof, justifying every step. You may use, without proof, the fact that $\sqrt{2} \notin \mathbb{Q}$. You have to prove any other property of irrational numbers that you wish to use.

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Label any parts that you want marked with the appropriate question and part number.

Question 7. [20 MARKS]

Prove or disprove the following claim:

$$\forall x \in \mathbb{Z}, [(\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)]$$

Give a detailed structured proof, justifying every step.

Only write your name on this page.

Family Name(s): ______________________________

Given Name(s): ______________________________

Total Marks = 90