Prove or disprove each of the following statements. Write detailed proof structures and justify your work.

1. For all real numbers \( r, s \), if \( r \) and \( s \) are both positive, then \( \sqrt{r} + \sqrt{s} \neq \sqrt{r+s} \).

**First, write the statement symbolically:**

\[
\forall r \in \mathbb{R}, \forall s \in \mathbb{R}, r > 0 \land s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s}
\]

**Second, try a direct proof:**

Assume \( r \in \mathbb{R} \) and \( s \in \mathbb{R} \)

Assume \( r > 0 \) and \( s > 0 \)

Then, \( \sqrt{r} + \sqrt{s} = \ldots \) No obvious way to continue.

**Next, try an indirect proof:**

Assume \( r \in \mathbb{R} \) and \( s \in \mathbb{R} \).

Assume \( \sqrt{r} + \sqrt{s} = \sqrt{r+s} \).

Then, \((\sqrt{r} + \sqrt{s})^2 = (\sqrt{r+s})^2 \). \# square both sides

Then, \((\sqrt{r})^2 + 2\sqrt{rs} + (\sqrt{s})^2 = r + s \). \# expand both sides

Then, \(2\sqrt{rs} = 0 \). \# subtract \( r + s \) from both sides

Then, \( rs = 0 \). \# divide by 2 and square both sides

Then, \( r = 0 \lor s = 0 \).

\# Now, do a sub-proof by cases.

Assume \( r = 0 \).

Then, \( r \neq 0 \).

Then, \( r \neq 0 \lor s \neq 0 \).

Then, \( \neg (r > 0 \land s > 0) \).

Assume \( s = 0 \).

Then, \( s \neq 0 \).

Then, \( r \neq 0 \lor s \neq 0 \).

Then, \( \neg (r > 0 \land s > 0) \).

In either case, \( \neg (r > 0 \land s > 0) \).

Then, \( \sqrt{r} + \sqrt{s} = \sqrt{r+s} \Rightarrow \neg (r > 0 \land s > 0) \)

Then, \( r > 0 \land s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s} \).

Then, \( \forall r \in \mathbb{R}, \forall s \in \mathbb{R}, r > 0 \land s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s} \).
2. For all real numbers $x$ and $y$, $x^4 + x^3 y - xy^3 - y^4 = 0$ exactly when $x = \pm y$.

**First, write the statement symbolically (be careful to handle that “±” correctly):**

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^4 + x^3 y - xy^3 - y^4 = 0 \iff (x = y \lor x = -y)$$

**Second, start the proof structure for the universal quantifiers:**

Assume $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

`# To prove an equivalence, we prove the implication in each direction.`

First assume $x^4 + x^3 y - xy^3 - y^4 = 0$.

Then, $x^3(x + y) - y^3(x + y) = 0$. # factor out the expression

Then, $(x^3 - y^3)(x + y) = 0$. # factor out the expression

Then, $x^3 - y^3 = 0 \lor x + y = 0$. # $ab = 0 \iff a = 0 \lor b = 0$

`# Now, do a sub-proof by cases.`

Assume $x^3 - y^3 = 0$

Then, $x^3 = y^3$. # add $y^3$ to both sides

Then, $x = y$ # take cube roots on both sides, cube root is one-to-one so we can do it

Then, $x = y \lor x = -y$ # introduce $\lor$

Assume $x + y = 0$

Then, $x = -y$ # subtract $y$ from both sides

Then, $x = y \lor x = -y$ # introduce $\lor$

In either case, $x = y \lor x = -y$.

Then, $x^4 + x^3 y - xy^3 - y^4 = 0 \implies x = \pm y$.

Next assume $x = \pm y$.

Then, $x = y \lor x = -y$. # expand “±”

`# Now, do a sub-proof by cases.`

Assume $x = y$.

Then, $x^3 = y^3$. # cube both sides

Then, $x^3 - y^3 = 0$. # subtract $y^3$ from both sides

Then, $(x^3 - y^3)(x + y) = 0$. # multiply both sides by $(x + y)$

Then, $x^4 + x^3 y - xy^3 - y^4 = 0$. # expand

Assume $x = -y$.

Then, $x + y = 0$. # add $y$ to both sides

Then, $(x^3 - y^3)(x + y) = 0$. # multiply both sides by $(x^3 - y^3)$

Then, $x^4 + x^3 y - xy^3 - y^4 = 0$. # expand

In both cases, $x^4 + x^3 y - xy^3 - y^4 = 0$.

Then, $x = \pm y \implies x^4 + x^3 y - xy^3 - y^4 = 0$.

Then, $x^4 + x^3 y - xy^3 - y^4 = 0 \iff x = \pm y$. # introduce $\iff$

Then, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^4 + x^3 y - xy^3 - y^4 = 0 \iff (x = y \lor x = -y)$.

Notice how the detailed proof structure makes it easy to keep track of assumptions, and cases and sub-cases, and to know exactly when we are done.