1. For each equivalence below, either provide a derivation from one side of the equivalence to the other (justify each step of your derivation with a brief explanation — for example, by naming one of the equivalences (see over for a list), or show that the equivalence does not hold (warning: you cannot use a derivation to show non-equivalence — instead, think carefully about what an equivalence means, and how you can disprove it).

   (a) $(P \Rightarrow R) \land (Q \Rightarrow R) \iff (P \lor Q) \Rightarrow R$
   **Sample Solution:**

$$\begin{aligned}
(P \Rightarrow R) \land (Q \Rightarrow R) &\iff (\neg P \lor R) \land (\neg Q \lor R) && \text{(implication)}\\
&\iff (\neg P \land \neg Q) \lor R && \text{(distributivity)}\\
&\iff \neg(P \lor Q) \lor R && \text{(DeMorgan's law)}\\
&\iff (P \lor Q) \Rightarrow R && \text{(implication)}
\end{aligned}$$

   (b) $P \Rightarrow (Q \Rightarrow R) \iff (P \Rightarrow Q) \Rightarrow R$
   **Sample Solution:**

   The equivalence does not always hold:
   Let $P = \mathsf{False}$, $Q = \mathsf{True}$, $R = \mathsf{False}$.
   Then $[P \Rightarrow (Q \Rightarrow R)] = [\mathsf{False} \Rightarrow (\mathsf{True} \Rightarrow \mathsf{False})] = [\mathsf{False} \Rightarrow \mathsf{False}] = \mathsf{True}$ and $[(P \Rightarrow Q) \Rightarrow R] = [(\mathsf{False} \Rightarrow \mathsf{True}) \Rightarrow \mathsf{False}] = [\mathsf{True} \Rightarrow \mathsf{False}] = \mathsf{False}$.

   (c) $P \Rightarrow (Q \Rightarrow R) \iff (P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$
   **Sample Solution:**

$$\begin{aligned}
(P \Rightarrow Q) \Rightarrow (P \Rightarrow R) &\iff \neg(P \Rightarrow Q) \lor (\neg P \lor R) && \text{(implication)}\\
&\iff (P \land \neg Q) \lor (\neg P \lor R) && \text{(implication negation)}\\
&\iff \big((P \lor \neg P) \land (\neg Q \lor \neg P)\big) \lor R && \text{(distributivity)}\\
&\iff (\neg P \lor \neg Q) \lor R && \text{(identity and commutativity)}\\
&\iff \neg P \lor (\neg Q \lor R) && \text{(associativity)}\\
&\iff P \Rightarrow (Q \Rightarrow R) && \text{(implication)}
\end{aligned}$$

**Standard Equivalences** (where $P$, $Q$, $P(x)$, $Q(x)$, *etc.* are arbitrary sentences)

- *Commutativity*
  $P \wedge Q \iff Q \wedge P$
  $P \vee Q \iff Q \vee P$
  $P \Leftrightarrow Q \iff Q \Leftrightarrow P$
- *Associativity*
  $P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge R$
  $P \vee (Q \vee R) \iff (P \vee Q) \vee R$
- *Identity*
  $P \wedge (Q \vee \neg Q) \iff P$
  $P \vee (Q \wedge \neg Q) \iff P$
- *Absorption*
  $P \wedge (Q \wedge \neg Q) \iff Q \wedge \neg Q$
  $P \vee (Q \vee \neg Q) \iff Q \vee \neg Q$
- *Idempotency*
  $P \wedge P \iff P$
  $P \vee P \iff P$
- *Double Negation*
  $\neg\neg P \iff P$
- *DeMorgan's Laws*
  $\neg(P \wedge Q) \iff \neg P \vee \neg Q$
  $\neg(P \vee Q) \iff \neg P \wedge \neg Q$
- *Distributivity*
  $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
  $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
- *Implication*
  $P \Rightarrow Q \iff \neg P \vee Q$
- *Biconditional*
  $P \Leftrightarrow Q \iff (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- *Renaming* (where $P(x)$ does not contain variable $y$)
  $\forall x, P(x) \iff \forall y, P(y)$
  $\exists x, P(x) \iff \exists y, P(y)$
- *Quantifier Negation*
  $\neg\forall x, P(x) \iff \exists x, \neg P(x)$
  $\neg\exists x, P(x) \iff \forall x, \neg P(x)$
- *Quantifier Commutativity*
  $\forall x, \forall y, S(x,y) \iff \forall y, \forall x, S(x,y)$
  $\exists x, \exists y, S(x,y) \iff \exists y, \exists x, S(x,y)$
- *Quantifier Distributivity* (where $S$ does not contain variable $x$)
  $S \wedge \forall x, Q(x) \iff \forall x, S \wedge Q(x)$
  $S \vee \forall x, Q(x) \iff \forall x, S \vee Q(x)$
  $S \wedge \exists x, Q(x) \iff \exists x, S \wedge Q(x)$
  $S \vee \exists x, Q(x) \iff \exists x, S \vee Q(x)$

2. An "interpretation" for a logical statement consists of a domain $D$ (any non-empty set of elements) and a meaning for each predicate symbol. For example, $D = \{1, 2\}$ and $P(x)$: "$x > 0$" is an interpretation for the statement $\forall x \in D, P(x)$ (in this case, one that happens to make the statement True). For each statement below, provide one interpretation under which the statement is true and another interpretation under which the statement is false — if either case is not possible, explain why clearly and concisely.

(a) $\forall x \in D, \forall y \in D, P(x, y)$

**Sample Solution:** *Let $D = \{1, 2\}$ and $P(x, y)$: "$x < y$." Then, $\forall x \in D, \forall y \in D, P(x, y)$ is False because $P(2, 1)$ is False.*
*Let $D = \{1\}$ and $P(x, y)$: "$x = y$." Then $\forall x \in D, \forall y \in D, P(x, y)$ is True because $P(1, 1)$ is True.*

(b) $(P \wedge Q) \Rightarrow (P \vee Q)$

**Sample Solution:** *Let $P =$ False and $Q =$ True. Then*

$$\begin{aligned}(P \wedge Q) \Rightarrow (P \vee Q) &= (\textit{False} \wedge \textit{True}) \Rightarrow (\textit{False} \vee \textit{True}) \\ &= \textit{False} \Rightarrow \textit{True} \\ &= \textit{True}\end{aligned}$$

*It is impossible to make the statement False because this would require $P \wedge Q$ to be True (meaning both $P$ and $Q$ are True) while at the same time $P \vee Q$ is False (meaning both $P$ and $Q$ are False).*

(c) $(\forall x \in D, \exists y \in D, P(x, y)) \Rightarrow (\exists y \in D, \forall x \in D, P(x, y))$

**Sample Solution:** *Let $D = \{1, 2\}$ and $P(x, y)$: "$x$ is an integer multiple of $y$." Then, $\forall x \in D, \exists y \in D, P(x, y)$ is True because $P(1, 1)$ and $P(2, 2)$ are both True (i.e., we can always pick $y = x$ for every $x$). Moreover, $\exists y \in D, \forall x \in D, P(x, y)$ is True because $P(1, 1)$ and $P(2, 1)$ are both True (i.e., $y = 1$ works for every $x$). Hence, the entire statement is True.*
*Let $D = \{1, 2\}$ and $P(x, y)$: "$x \neq y$." Then, $\forall x \in D, \exists y \in D, P(x, y)$ is True because $P(1, 2)$ and $P(2, 1)$ are both True (i.e., we can always pick $y$ to be different from $x$). However, $\exists y \in D, \forall x \in D, P(x, y)$ is False because there is no value of $y \in D$ that is different from every other element of $D$. Hence, the entire statement is False.*