CSC165, Summer 2014
Lecture notes: Week 4

1 7/2 is not a natural number

We would like to prove that $7/2 \notin \mathbb{N}$. To do that we need to first convince ourselves that that's true, and then to find a way to connect our intuition to the definition of $\mathbb{N}$.

Here's the intuition: 7/2 lies *between* 3 and 4, and there are no natural numbers between 3 and 4.

Here's the definition of $\mathbb{N}$:

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$

First, let's solve a simpler problem: proving that $7/2 \notin \{0, 1, 2, 3, 4, 5\}$. Since the set is finite, we can just check that $7 \neq 2 * 0 = 0$, $7 \neq 2 * 1 = 2$, $7 \neq 2 * 2 = 4$, $7 \neq 2 * 3 = 6$, $7 \neq 2 * 4 = 8$, $7 \neq 2 * 5 = 10$. That completes the proof.

We can't use the same technique to prove that $7/2 \notin \mathbb{N}$, since we can't check that 7/2 isn't equal to infinitely many numbers!

So now let's use our intuition to prove that $7/2 \notin \{0, 1, 2, 3, ...\}$ without exhaustively checking every number.

We need to show that 7/2 lies between 3 and 4. Here's how:

Let's say $7/2 < k \Leftrightarrow 7 < 2k$. Then for $m > k$, $7 < 2m \Leftrightarrow 7/2 < m \Rightarrow 7/2 \neq m$ since $2m > 2k$. Now $7 < 2 * 4 \Leftrightarrow 7/2 < 4$, so 7/2 can't be larger than 4.

Let's say $7/2 > k \Leftrightarrow 7 > 2k$. Then for $m < k$, $7 > 2m \Leftrightarrow 7/2 > m \Rightarrow 7/2 \neq m$ since $2m < 2k$. Now $7 > 2 * 3 \Leftrightarrow 7/2 > 3$, so 7/2 can't be smaller than 3.

We can check that $7/2 \neq 3$ and $7/2 \neq 4$, so the only option left is for 7/2 to be between 3 and 4! We can now go back to the definition of $\mathbb{N}$ and observe that the naturals are listed in increasing order, so if we don't see any 7/2 between 3 and 4 in the list of all the naturals, we don't need to look anywhere else, since we've proven that 7/2 can't be to the right of 4 or to the left of 3.

So we've proven that $7/2 \notin \mathbb{N}$.

This might seem like too much work to prove such an seemingly obvious statement, but you can't be sure of something until you've proven it!