CSC165, Summer 2014
Supplementary notes: Week 1

1 $\sqrt{2}$ is irrational

$\mathbb{R}$ is the set of all real numbers. There are several ways to define real numbers. One way is to say that $\mathbb{R}$ is the set of all numbers that can be written as infinite decimal fractions (e.g., 1.0000..., 5.25000000000..., 100.112123123412345123456..., π).

Basically, rational numbers are all the fractions (this includes fractions like $-5/1$, i.e., whole numbers). Formally, we can define rational numbers like this:

Definition 1. *x is **rational** if $x = p/q$ for some whole number p and natural number q.*

We denote the set of rational numbers by $\mathbb{Q}$ (for *quotient*).

Definition 2. *x is **irrational** if $x \in \mathbb{R}$ and $x \notin \mathbb{Q}$.*

Here's how we can prove that $\sqrt{2}$ is irrational. Suppose $\sqrt{2}$ is rational. Then there are whole numbers p and q such that $\sqrt{2} = p/q$, and we can assume that p and q do not have common divisors since you can always reduce a fraction. In other words, $q = q_1 q_2 .... q_n$, $p = p_1 p_2 .... p_m$ where the q_i and the p_i are prime and $q_i \neq p_j$ always. But $2 = p^2/q^2$ so that p^2 and q^2 have a common factor, namely q^2! But if p and q don't have common factors, then p^2 and q^2 don't either, since the prime factors of p^2 and q^2 are just $\{p_1, p_2, ...p_n\}$ and $\{q_1, q_2, ...q_n\}$. So if we assume that $\sqrt{2} = p/q$ where p and q do not have a common factor, we conclude that p^2 and q^2 both do and don't have a common factor! So our assumption must have been wrong. So it can't be the case that $\sqrt{2} = p/q$ where p and q do not have a common factor. So it can't be the case that $\sqrt{2} = p/q$ in general for whole p and q. So $\sqrt{2}$ is not a rational number.