CSC165, Summer 2014
A Primer on Proving Inequalities

1 Introduction

In this primer, we describe some elementary techniques for proving inequalities. Here are some properties of inequalities (some of those are taken from Section 1.5 of the notes): For anye $x \in \mathbb{R}, y \in \mathbb{R}, w \in \mathbb{R}, z \in \mathbb{R}$:

$$(x < y) \wedge (w \leqslant z) \Rightarrow [x + w < y + z] \tag{1}$$
$$(x < y) \wedge (z > 0) \Rightarrow [xz < yz] \tag{2}$$
$$(x < y) \wedge (z < 0) \Rightarrow [xz > yz] \tag{3}$$
$$(x < y) \wedge (y \leqslant z) \Rightarrow [x < z] \tag{4}$$
$$(x \leqslant y) \wedge (y < z) \Rightarrow [x < z] \tag{5}$$
$$(x \leqslant y) \wedge (y \leqslant z) \Rightarrow [x \leqslant z] \tag{6}$$
$$|x + y| \leqslant |x| + |y| \tag{7}$$
$$x^2 \geqslant 0 \tag{8}$$

2 $(x < y) \wedge (z > 0) \Rightarrow [xz < yz]$, $(x < y) \wedge (z < 0) \Rightarrow [xz > yz]$

Basically, multiplying (or dividing) by a positive number preserves the inequality. Multiplying (or dividing) by a negative number "flips" the inequality.

For example, for real numbers a and b,

$$a > b \Leftrightarrow -b < -a,$$

since we transform the inequality by multiplying it by -1, which is a negative number (both going left-to-right and right-to-left).

For a positive natural number n,

$$n^2 > 1 \Leftrightarrow n^3 > n,$$

since we transform the inequality by multiplying it by n (going left-right) or dividing it by n (going right-to-left), and n is a positive number by assumption[1]

Keep in mind that if z is 0, the inequality is not preserved: $1 < 2$, but it is not the case that $0 * 1 = 0 * 2$. It is, however, true that $(x < y) \wedge (z \geqslant 0) \Rightarrow [xz \leqslant yz]$, with $xz = yz$ happening when $z = 0$.

[1]Strictly speaking the assumption that n is positive (i.e., not zero) is not needed there, since it follows from both $n^2 > 1$ and $n^3 > n$.

3 $(x < y) \wedge (w \leqslant z) \Rightarrow [x + w < y + z]$

First, it's important to have the intuition for why this is true. Imagine you're starting with

$$x < y.$$

Now if we add the smaller value (w) to the left-hand side (LHS), which is smaller to begin with, and the larger (or equal to w) value (z) to the right-hand side (RHS), then the inequality still holds (i.e., $[x + w < y + z]$ is true).

Suppose that we would like to prove that, for any $n > 1$,

$$6n^3 > 3n + 2.$$

The idea is to use that for any $n > 1$, $3n^3 > 3n$ and $2n^3 > 2$.

First, we prove that for any natural $n > 1$ and any positive real a, $an^3 > an$:

Assume $n \in \mathbb{N}, a \in \mathbb{R}^{\geqslant 0}$
  Assume $n > 1$
    Then $n^2 = n * n > n * 1 = n > 1$ # multiplication by a positive number n, algebra, assumption that $n > 1$
    Then $n^2 > 1$ # omit intermediate terms in the line above
    Also $an > a * 1 > 0$ # multiply both sides of an inequality by $3 > 0$
    Then $an^3 > an$ # multiply both sides of an inequality by $3n > 0$
  Then $[n > 1] \Rightarrow an^3 > an$ # introduce implication
Then $\forall n \in \mathbb{N}, \forall a \in \mathbb{R}^{\geqslant 0}, [n > 1] \Rightarrow an^3 > an$ # introduce universal twice

Note that in order to obtain this proof, we could start with the the thing we're trying to prove, and "work backwards:" Start at $an^3 > an$, and then divide by an to get $n^2 > 1$. When presenting the proof, we first prove $n^2 > 1$ (which is true since $n > 1$), and then get to $3n^3 > 3n$, but we know where to start because we started with what we wanted to prove and then worked backwards.

You can similarly prove that $2n^3 > 2$ for $n > 1$.

Now we can prove that for any $n > 1$, $6n^3 > 3n + 2$. We start at what we're trying to prove and work backwards to obtain the proof. But be careful: when working backwards, we need to make sure that the derivation will actually be correct when presenting it going "forward."

We start at $6n^3 > 3n + 2$. This is implied by $n^3 + 3n^3 + 2n^3 > 3n + 2$ (we have just re-expressed the LHS.) This is implied by $n^3 + 2n^3 > 2$ since for $n > 1$ since we proved that $an^3 > an$ for positive a (and we can set $a = 3$). (Note that it's not true that for $n > 1$, $n^3 + 3n^3 + 2n^3 > 3n + 2 \Rightarrow n^3 + 2n^3 > 2$, but it is true that for $n > 1$, $n^3 + 2n^3 > 2 \Rightarrow n^3 + 3n^3 + 2n^3 > 3n + 2$. The direction in which we work is important.) Now $n^3 + 2n^3 > 2$ is implied by $n^3 > 0$ since we can prove (but haven't proven here that $2n^3 > 2$ for $n > 1$. $n^3 > 0$ is true since $n > 0$, and so $n * n^2 > 0 * n^2$ since $n^2 > 0$.

We now prove the statement by working backwards.

Assume $n \in \mathbb{N}$
  Assume $n > 1$
    Then $n > 0$ # $n > 1 > 0$
    Then $n^2 * n > n^2 0$ # $n^2 > 0$
    Then $n^3 > 0$ # algebra
    Then $n^3 + 3n^3 > 3n$ # $3n^3 > 3n$ for $n > 1$ proved above, $n > 1$ by assumption, $(x < y) \wedge (z > 0) \Rightarrow [xz < yz]$
    Then $n^3 + 3n^3 + 2n^2 > 3n + 2$ # $2n^2 > 2$ for $n > 1$ proved by the reader, $n > 1$ by assumption, $(x < y) \wedge (z > 0) \Rightarrow [xz < yz]$
    Then $6n^3 > 3n + 2$ # algebra
  Then $[n > 1] \Rightarrow 6n^3 > 3n + 2$ # introduce implication
Then $\forall n \in \mathbb{N}, [n > 1] \Rightarrow 6n^3 > 3n + 2$ # introduce universal

4 $x^2 \geqslant 0$

The square of any real number is non-negative. Why is this true? One way to see it is to say that $x^2 = x * x$, so it's either a product of two negative numbers or a product of two non-negative numbers. Either way, it's non-negative.

The property $x^2 \geqslant 0$ is sometimes useful in itself. Here's another instance where it's useful. To prove that $x^2 - 2x + 2 > 0$, we might say $x^2 - 2x + 2 = x^2 - 2x + 1 + 1 = (x-1)^2 + 1 > 0$, where the last step is justified by starting with $1 > 0$, and then adding $(x-1)^2$ to the LHS and 0 to the RHS.