CSC165 Final Exam Aid Sheet and Draft Paper: PRINT SEPARATELY

1 Basic laws of manipulating formal statements

identity laws	$P \wedge (Q \vee \neg Q)$	$\iff$	P
	$P \vee (Q \wedge \neg Q)$	$\iff$	P
idempotency laws	$P \wedge P$	$\iff$	P
	$P \vee P$	$\iff$	P
commutative laws	$P \wedge Q$	$\iff$	$Q \wedge P$
	$P \vee Q$	$\iff$	$Q \vee P$
	$(P \Leftrightarrow Q)$	$\iff$	$(Q \Leftrightarrow P)$
associative laws	$(P \wedge Q) \wedge R$	$\iff$	$P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R$	$\iff$	$P \vee (Q \vee R)$
distributive laws	$P \wedge (Q \vee R)$	$\iff$	$(P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R)$	$\iff$	$(P \vee Q) \wedge (P \vee R)$
contrapositive	$P \Rightarrow Q$	$\iff$	$\neg Q \Rightarrow \neg P$
implication	$P \Rightarrow Q$	$\iff$	$\neg P \vee Q$
equivalence	$(P \Leftrightarrow Q)$	$\iff$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
double negation	$\neg(\neg P)$	$\iff$	P
DeMorgan's laws	$\neg(P \wedge Q)$	$\iff$	$\neg P \vee \neg Q$
	$\neg(P \vee Q)$	$\iff$	$\neg P \wedge \neg Q$
implication negation	$\neg(P \Rightarrow Q)$	$\iff$	$P \wedge \neg Q$
equivalence negation	$\neg(P \Leftrightarrow Q)$	$\iff$	$\neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)$
quantifier negation	$\neg(\forall x \in D, P(x))$	$\iff$	$\exists x \in D, \neg P(x)$
	$\neg(\exists x \in D, P(x))$	$\iff$	$\forall x \in D, \neg P(x)$
quantifier distributive laws	$\forall x \in D, P(x) \wedge Q(x)$	$\iff$	$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$
(where R does not contain variable x)	$\exists x \in D, P(x) \vee Q(x)$	$\iff$	$(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$
	$\forall x \in D, R \wedge Q(x)$	$\iff$	$R \wedge (\forall x \in D, Q(x))$
	$\forall x \in D, R \vee Q(x)$	$\iff$	$R \vee (\forall x \in D, Q(x))$
	$\exists x \in D, R \vee Q(x)$	$\iff$	$R \vee (\exists x \in D, Q(x))$
	$\exists x \in D, R \wedge Q(x)$	$\iff$	$R \wedge (\exists x \in D, Q(x))$
variable renaming	$\forall x \in D, P(x)$	$\iff$	$\forall y \in D, P(y)$
(where y does not appear in $P(x)$)	$\exists x \in D, P(x)$	$\iff$	$\exists y \in D, P(y)$

2 Rules of inference

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Introduction rules

[¬I] negation introduction

Assume A

$\vdots$

contradiction

$$\frac{\text{contradiction}}{\neg A}$$

[∧I] conjunction introduction

$$\frac{A \quad B}{A \wedge B}$$

[∨I] disjunction introduction

$$\frac{A}{A \vee B \quad B \vee A} \qquad \frac{}{A \vee \neg A}$$

[⇒I] implication introduction

(direct)

Assume A

$\vdots$

$$\frac{B}{A \Rightarrow B}$$

(indirect)

Assume $\neg B$

$\vdots$

$$\frac{\neg A}{A \Rightarrow B}$$

[⇔I] equivalence/bi-implication introduction

$$\frac{A \Rightarrow B \quad B \Rightarrow A}{A \Leftrightarrow B}$$

[∀I] universal introduction

Assume $a \in D$

$\vdots$

$$\frac{P(a)}{\forall x \in D, P(x)}$$

[∃I] existential introduction

$$\frac{P(a) \quad a \in D}{\exists x \in D, P(x)}$$

Elimination rules

[¬E] negation elimination

$$\frac{\neg\neg A}{A} \qquad \frac{A \quad \neg A}{\text{contradiction}}$$

[∧E] conjunction elimination

$$\frac{A \wedge B}{A \quad B}$$

[∨E] disjunction elimination

$$\frac{A \vee B \quad \neg A}{B} \qquad \frac{A \vee B \quad \neg B}{A}$$

[⇒E] implication elimination

(Modus Ponens)

$$\frac{A \Rightarrow B \quad A}{B}$$

(Modus Tollens)

$$\frac{A \Rightarrow B \quad \neg B}{\neg A}$$

[⇔E] equivalence/bi-implication elimination

$$\frac{A \Leftrightarrow B}{A \Rightarrow B \quad B \Rightarrow A}$$

[∀E] universal elimination

$$\frac{\forall x \in D, P(x) \quad a \in D}{P(a)}$$

[∃E] existential elimination

$$\frac{\exists x \in D, P(x)}{\text{Let } a \in D \text{ such that } P(a)}$$

$\vdots$

3 Asymptotic notation

For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$ (i.e., any function mapping naturals to nonnegative reals), let

$$\mathcal{O}(f) = \left\{ g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \leqslant cf(n) \right\}.$$

4 Inequalities

For any $x \in \mathbb{R}, y \in \mathbb{R}, w \in \mathbb{R}, z \in \mathbb{R}$:

$$(x < y) \wedge (w \leqslant z) \Rightarrow [x + w < y + z] \tag{1}$$
$$(x < y) \wedge (z > 0) \Rightarrow [xz < yz] \tag{2}$$
$$(x < y) \wedge (z < 0) \Rightarrow [xz > yz] \tag{3}$$
$$(x < y) \wedge (y \leqslant z) \Rightarrow [x < z] \tag{4}$$
$$(x \leqslant y) \wedge (y < z) \Rightarrow [x < z] \tag{5}$$
$$(x \leqslant y) \wedge (y \leqslant z) \Rightarrow [x \leqslant z] \tag{6}$$
$$|x + y| \leqslant |x| + |y| \tag{7}$$
$$x^2 \geqslant 0 \tag{8}$$

5 Induction

The uses of the Principle of Simple Induction are as follows.

For any predicate P,

$$[P(0) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))] \Rightarrow [\forall n \in \mathbb{N}, P(n)]$$

More generally, for any predicate P and $k \in \mathbb{N}$,

$$[P(k) \wedge (\forall n \in \{k, k+1, k+2, ...\}, P(n) \Rightarrow P(n+1))] \Rightarrow [\forall n \in \{k, k+1, k+2, ...\}, P(n)]$$