You are allowed to use (i.e., put in your justifications), without proof, the following facts:

- $\forall x \in \mathbb{R}, x^2 \geq 0$
- For $a, b, c$ real, $a > 0$, $b^2 - 4ac < 0 \Rightarrow \forall x, ax^2 + bx + c \geq 0$
- $f(n) = 1/n$ is a decreasing function for $n > 0$. In other words, $\forall n_1 \in \mathbb{N}, \forall n_2 \in \mathbb{N}, n_2 > n_1 \Rightarrow 1/n_2 < 1/n_1$

1. Prove or disprove using a detailed structured proof, justifying every step:

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, [x(x + 1) = y(y + 1)] \iff [x = y]$$

**Solution:** The statement is false.

**Proof:**

Let $x_0 = -1, y_0 = 0$
Then $x_0 \in \mathbb{Z}, y_0 \in \mathbb{Z}$
Then $x_0(x_0 + 1) = 0$  # $x_0 + 1 = 0$
Then $y_0(y_0 + 1) = 0$  # $y_0 = 0$
Then $x_0(x_0 + 1) = y_0(y_0 + 1)$  # both equal 0
Then also $\neg[x_0 = y_0]$  # $-1 \neq 0$
Then $[x(x + 1) = y(y + 1)] \land \neg[x = y]$  # conjunction of two true statements
Then $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, [x(x + 1) = y(y + 1)] \land \neg[x = y]$  # $x_0, y_0$ are such $x, y$
Then $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \neg([x(x + 1) = y(y + 1)] \Rightarrow [x = y])$  # implication negation
Then $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, [\neg([x(x + 1) = y(y + 1)] \Rightarrow [x = y])] \lor [\neg([x(x + 1) = y(y + 1)] \Leftarrow [x = y])]$  # True $\lor$ R is True for any R
Then $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \neg([x(x + 1) = y(y + 1)] \Leftarrow [x = y])$  # equivalence negation
Then $\exists x \in \mathbb{Z}, \neg\forall y \in \mathbb{Z}, [x(x + 1) = y(y + 1)] \Leftarrow [x = y]$  # quantifier negation
Then $\neg[\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, [x(x + 1) = y(y + 1)] \Leftarrow [x = y]]$  # quantifier negation

Note: less detailed arguments that still end with the same conclusion may be fine.
2. Prove or disprove using a detailed structured proof, justifying every step:
\[ \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [(x^2y < 0) \land (x \neq 0)] \Leftrightarrow [x^{100}y < 0] \]

**Solution:** The statement is true.

**Proof:**
Assume \( x \in \mathbb{R}, y \in \mathbb{R} \)
Then \( x^{49} \in \mathbb{R} \) \# the reals are closed under multiplication and \( x^{49} = x \cdot x \cdot x \ldots \cdot x \)
Also \( x^{49} \neq 0 \) \# \( \forall x \in \mathbb{R}, x = 0 \Leftrightarrow x^n = 0 \)
Then \( (x^{49})^2 > 0 \) \# \( \forall a \in \mathbb{R}, a \neq 0 \Rightarrow a^2 > 0 \)
Then \( x^{98} > 0 \) \# algebra

Assume \( (x^2y < 0) \land (x \neq 0) \)
Then \( x^{98}(x^2y < 0) \) \# \( \forall a \in \mathbb{R}, \forall b \in \mathbb{R}, [a > 0, b < 0] \Rightarrow ab < 0 \)
Then \( x^{100}y < 0 \) \# algebra/associativity of multiplication

Then \([x^2y < 0) \land (x \neq 0)] \Rightarrow [x^{100}y < 0] \) \# assuming the antecedent leads to the consequent

Assume \( x^{100}y < 0 \)
Then \( x^{98}(x^2y) < 0 \) \# algebra/associativity of multiplication
Also \( 1/x^{98} > 0 \) \# \( \forall a \in \mathbb{R}, a > 0 \Rightarrow 1/a > 0 \) Then \( (1/x^{98})x^{98}(x^2y) < (1/x^{98})0 \) \# multiply both sides by the same value
Then \( x^2y < 0 \) \# algebra

Then \([x^{100}y < 0] \Rightarrow [(x^2y < 0) \land (x \neq 0)] \) \# assuming the antecedent leads to the consequent
Then \([x^2y < 0) \land (x \neq 0)] \Leftrightarrow [x^{100}y < 0] \) \# equivalence
Then \( \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [(x^2y < 0) \land (x \neq 0)] \leftrightarrow [x^{100}y < 0] \) \# introduce universal

3. As of Jun.17, I predict that Brazil, Germany, Italy, and Argentina will make it to the semi-finals of the FIFA World Cup. Suppose you have a group of 25 footballers from those four countries. Prove that out of those 25, there is a group of at least 6 footballers who come from the same country. *It is not necessary to use the detailed structured proof format, but you still must justify every step.*

**Solution:** The statement is true.

**Proof:**
Let \( N_c \) be the number of footballers in the group who come from country \( c \in \{\text{Brazil, Italy, Argentina, Germany}\} \)
Let \( N \) be the total number of footballers in the group
Assume there is no group of 6 footballers who come from the same country and \( N = 25 \)
Then \( \forall c \in \text{Brazil, Italy, Argentina, Germany}, N_c \leq 5 \) \# By assumption, the natural number \( N_c \) is smaller than 6
Then \( N = N_{\text{Brazil}} + N_{\text{Italy}} + N_{\text{Argentina}} + N_{\text{Germany}} \leq 5 + 5 + 5 + 5 = 20 \) \# Only footballers from the 4 countries are in the group + substitution
But it is not the case that \( 25 \leq 20 \) \# arithmetic, we assumed that \( N = 25 \)
Contradiction!
Then there is a group of at least 6 footballers who come from the same country \# assuming otherwise leads to a contradiction

\[ \]
4. Prove or disprove the following equivalence. If you decide it is true and so need to prove it, do not use a truth table. Justify every step in your derivation using the equivalences in Section 2.17 of the notes.

\[ \left( \neg Q \land P \right) \lor \neg Q \iff \left( \neg Q \land \neg R \right) \lor \left( \neg Q \land R \right) \]

**Solution:** The statement is true.

Proof:

\[
\left( \neg Q \land P \right) \lor \neg Q
\iff \left( \neg Q \land P \right) \lor \left( \neg Q \land \neg \left( P \lor \neg P \right) \right) \quad \# \text{identity laws}
\iff \left( \neg Q \land P \right) \lor \left( \neg Q \land \neg Q \right) \quad \# \text{distributive laws}
\iff \left( \neg Q \land P \right) \lor \left( \neg Q \land \neg Q \right) \quad \# \text{idempotency laws}
\iff \left( \neg Q \land \left( P \lor \neg P \right) \right) \quad \# \text{distributive laws}
\iff \neg Q \quad \# \text{identity laws}
\iff \neg Q \land \left( \neg R \lor R \right) \quad \# \text{identity laws}
\iff \left( \neg Q \land \neg R \right) \lor \left( \neg Q \land R \right) \quad \# \text{distributive laws}
\]

5. Let \( R^+ \) be the set of positive real numbers. Prove or disprove using a detailed structured proof, justifying every step:

\[ \exists \epsilon \in R^+, \forall B \in N, \forall n \in N, [n > B] \implies \left| \frac{2n^2 + 15n}{n^2} - 2 \right| < \epsilon \]

**Solution:** The statement is true.

Proof:

Let \( \epsilon_0 = 16 \)

Assume \( B \in N, n \in N \)

Assume \( n > B \)

Then \( B \geq 0 \) \quad \# The smallest natural number is zero

Then \( n \geq 1 \) \quad \# \quad n > B \geq 0

Then \( \left| \frac{2n^2 + 15n}{n^2} - 2 \right| = \frac{15}{n^2} \leq 15 \) \quad \# Algebra, \( \forall n_1 \in N, \forall n_2 \in N, n_2 > n_1 \implies 1/n_2 < 1/n_1 \) applied to \( n_1 = 1 \)

Then \( \left| \frac{2n^2 + 15}{n^2} - 2 \right| < \epsilon_0 \) \quad \# \quad 16 > 15

Then \( \left[ n > B \right] \quad \# \quad \text{assuming the antecedent implies the consequent} \)

Then \( \forall B \in N, \forall n \in N, [n > B] \implies \left| \frac{2n^2 + 15}{n^2} - 2 \right| < \epsilon_0 \) \quad \# \quad \text{introduce universal}

Then \( \exists \epsilon \in R^+, \forall B \in N, \forall n \in N, [n > B] \implies \left| \frac{2n^2 + 15}{n^2} - 2 \right| < \epsilon \) \quad \# \quad \epsilon_0 = 16 \) is such an \( \epsilon \)

\( \Box \)
6. Assume that $D \subset \mathbb{N}$ and $D \neq \emptyset$. Prove or disprove using a detailed structured proof, justifying every step:

$[\forall x \in D, \exists y \in \mathbb{N}, y < x] \iff [0 \notin D]$

**Solution:** The statement is **true**.

**Proof:**

Assume $0 \in D$

Let $x_0 = 0$

Then $x_0 \in D$

Then $\forall y \in \mathbb{N}, y \geq x_0$ # $0$ is the smallest natural number

Then $\exists x \in D, \forall y \in \mathbb{N}, y \geq x$ # $x_0$ is such an $x$

Then $\neg(\forall x \in D, \exists y \in \mathbb{N}, y < x)$ # quantifier negation

Then $0 \in D \Rightarrow \neg(\forall x \in D, \exists y \in \mathbb{N}, y < x)$ # assuming the antecedent leads to the consequent

Then $\forall x \in D, \exists y \in \mathbb{N}, y < x \Rightarrow 0 \notin D$ # contrapositive

Assume $0 \notin D$

Assume $x \in D$

Then $x \neq 0$ # $0 \notin D$ but $x \in D$ Let $y_0 = 0$

Then $y_0 \in \mathbb{N}$ # definition of naturals

Then $\forall x \in D, y_0 < x$ # the minimum natural number is $0$ and $0 \notin D$ so $x \neq 0$

Then $\exists y \in \mathbb{N}, y < x$ # $y_0$ is such a $y$

Then $\forall x \in D, \exists y \in \mathbb{N}, y < x$ # introduce universal

Then $[0 \notin D] \Rightarrow [\forall x \in D, \exists y \in \mathbb{N}, y < x]$ # assuming the antecedent implies the consequent

Then $\forall x \in D, \exists y \in \mathbb{N}, y < x \iff 0 \notin D$ # equivalence