Assignment 1
Due Thursday, 14th October, 2010 at 6:10pm in tutorial

On the cover page of your assignment, you must write and sign the following statement: “I have read and understood the policy on collaboration on homework stated on the course web page.” Without this signed statement your homework will not be marked.

1. (5 marks) Prove by induction that $\forall n \geq 0, n \in \mathbb{N}$

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$  

2. (10 marks) Let $T(n, m) = T(n, m-1) + T(n-1, m)$ and $T(s, 2) = T(2, s) = s$ for all $s \in \mathbb{N}$.

Use induction to prove that $T(n, m) \leq \left(\frac{n+m-1}{n-1}\right)$ for all $n, m \in \mathbb{N}$ such that $n, m \geq 2$.

[Hint: The trick to this question is coming up with the correct induction hypothesis.]

3. (10 marks) Recall the Fibonacci numbers,

$$f_{ib}(n) = \begin{cases} f_{ib}(n-1) + f_{ib}(n-2) & n \geq 2 \\ 1 & n = 1 \\ 0 & n = 0. \end{cases}$$

Prove that the Fibonacci numbers satisfy the following identities

$$f_{ib}(2n-1) = (f_{ib}(n))^2 + (f_{ib}(n-1))^2$$
$$f_{ib}(2n) = (f_{ib}(n))^2 + 2f_{ib}(n)f_{ib}(n-1)$$

for all natural numbers $n \geq 1$.

4. (10 marks) Consider a chocolate bar of dimensions $2 \times n$. How many different ways are there to split the bar up into $2 \times 1$ size pieces? For example, a $2 \times 2$ bar has 2 ways, a $2 \times 3$ bar has 3 ways, a $2 \times 4$ bar has 5 ways, etc.

Prove your answer using induction for all natural numbers $n > 1$.

5. (10 marks) Let $M$ be the smallest set of real-valued matrices such that

- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in M,$
- if $m_1, m_2 \in M$, then $m_1 \cdot m_2 \in M$, and
- if $m \in M$, $r \in \mathbb{R}$ and $r \neq 0$, then $m_r \in M$, where $m_r$ is obtained from $m$ by multiplying every entry in the first row of $m$ by $r$.  

Prove inductively that every matrix in $M$ is invertible (HINT: prove that the determinant is not $0$).

6. (10 marks)
Prove that the following program is correct with respect to the following Precondition/Postcondition pair.
(Definition of \texttt{div}: if $a, b$ are integers with $b > 0$, then $a \text{ div } b$ and $a \text{ mod } b$ are the unique integers such that $a = (a \text{ div } b) \cdot b + a \text{ mod } b$.)
(HINT: You may use without proof the fact that $y = \lfloor \sqrt{m} \rfloor$ if and only if $y^2 \leq m$ and $(y + 1)^2 > m$.)

\textbf{Precondition}: $m$ is an integer, $m \geq 0$.

\textbf{Postcondition}: The program returns $\lfloor \sqrt{m} \rfloor$.

```
sqrt(m) {
  if $m = 0$ then
    return 0
  else
    $x := \sqrt{m \text{ div } 4}$
    if $(2 \cdot x + 1) \cdot (2 \cdot x + 1) \leq m$ then
      return $2 \cdot x + 1$
    else
      return $2 \cdot x$
    end if
  end if
}
```

7. (15 marks) Prove that the following program is correct with respect to the following Precondition/Postcondition pair. (Intuitively, the program is testing whether an array is sorted.)

\textbf{Precondition}: $m$ is an integer, $m \geq 1$. $A$ is an integer array.

\textbf{Postcondition}: If for all $i, j$ such that $1 \leq i < j \leq m$ it is the case that $A[i] \leq A[j]$, then $\text{TRUE}$ is returned; otherwise $\text{FALSE}$ is returned.

```
k := 1
while $k < m$ and $A[k] \leq A[k + 1]$ do
  $k := k + 1$
end while
if $k = m$ then
  return \text{TRUE}
else
  return \text{FALSE}
end if
```