

# The adaptable choosability number grows with the choosability number

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## Abstract

The *adaptable choosability number* of a multigraph  $G$ , denoted  $ch_a(G)$ , is the smallest integer  $k$  such that every edge labeling of  $G$  and assignment of lists of size  $k$  to the vertices of  $G$  permits a list coloring of  $G$  in which no edge  $e = uv$  has both  $u$  and  $v$  colored with the label of  $e$ . We show that  $ch_a$  grows with  $ch$ , i.e. there is a function  $f$  tending to infinity such that  $ch_a(G) \geq f(ch(G))$ .

*Keywords:* adaptable coloring, list coloring

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## 1. Introduction

Hell and Zhu introduced the adaptable chromatic number in [11]. Given a multigraph whose edges are labeled from  $[k] = \{1, 2, \dots, k\}$ , the goal is to color the vertices with colors from  $[k]$  so that there is no edge  $e = uv$  such that  $u$  and  $v$  are both colored with the label of  $e$ . A vertex coloring which satisfies this property is called an *adaptable vertex coloring*. The *adaptable chromatic number* of a graph  $G$ , denoted  $\chi_a(G)$ , is the minimum number  $k$  such that *every* edge labeling of  $G$  from  $[k]$  permits an adaptable vertex coloring from  $[k]$ . It has been studied in [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15] (in some cases by a different name).

Note that every proper vertex coloring of a graph  $G$  is an adaptable vertex coloring for any edge labeling and thus  $\chi_a(G) \leq \chi(G)$ . The inequality is tight as there are infinite families of graphs where  $\chi_a(G) = \chi(G)$  [10, 11]. These

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parameters can also be far apart as there are infinite families of graphs where  $\chi_a(G) = \Theta\left(\sqrt{\chi(G)}\right)$  (for example, the complete graph [4]). This brings us to the following question proposed by Hell and Zhu in [11].

**Question.** Is there a function  $f$  tending to infinity such that  $\chi_a(G) \geq f(\chi(G))$ ?

As far as we know, the answer may be ‘yes’ with  $f(k) = \Theta\left(\sqrt{k}\right)$ ; i.e. the complete graph may be asymptotically extremal.

In this paper, we study adaptable list coloring, which is defined naturally in [12]: Given a multigraph  $G$ , the *adaptable choosability number*, denoted  $ch_a(G)$ , is the minimum number  $k$  such that every edge labeling of  $G$  and assignment to each vertex  $v$  of a list  $L(v)$  of size  $k$ , there is an adaptable coloring of  $G$  from these lists. As with  $\chi_a$ , it is trivial that  $ch_a(G) \leq ch(G)$ , where  $ch(G)$  is the choosability number. We answer the list coloring version of Hell and Zhu’s question.

**Theorem 1.1.** *There is a function  $h$  tending to infinity such that  $ch_a(G) \geq h(ch(G))$ .*

Our proof obtains  $h(k) = \Theta\left(\log^{1/5}k\right)$ , but we made no effort to optimize it. As far as we know, we can have  $h(k) = \Theta\left(\sqrt{k}\right)$ . We know, however, that  $h(k) = O\left(\sqrt{k}\right)$  since, like with  $\chi_a$ , the complete graph has  $ch_a(K_n) = \Theta\left(\sqrt{ch(K_n)}\right)$  [12, 14].

The proof of the theorem uses a probabilistic approach and takes advantage of the Chernoff bound [3]. Instead of using the original statement we use the (weaker) version found in [13].

**Chernoff Bound.** *For any  $0 \leq t \leq np$ ,*

$$\Pr(|BIN(n, p) - np| > t) < 2e^{-\frac{t^2}{3np}},$$

*where  $BIN(n, p)$  is the sum of  $n$  independent variables, each equal to 1 with probability  $p$  and 0 otherwise.*

## 2. Proof of Main Theorem

The proof of Theorem 1.1 closely follows the approach taken by Alon in [1] for a similar result on normal list coloring.

We start by proving the following theorem, where  $\delta(G)$  is the minimum degree of  $G$ .

**Theorem 2.1.** *There is a function  $g$  tending to infinity such that if  $H$  is a bipartite graph satisfying  $\delta(H) \geq d$  then  $ch_a(H) \geq g(d)$ .*

Theorem 1.1 easily follows from Theorem 2.1.

*Proof of Theorem 1.1.* We use the following two well known and easily proved facts:

- (i) If  $\delta(G) \geq d$ ,  $G$  has a bipartite subgraph with minimum degree at least  $\frac{d}{2}$ . This can be seen by taking a spanning bipartite subgraph of  $G$  with the maximum number of edges.
- (ii) If  $ch(G) \geq k$ , then it has a subgraph of minimum degree at least  $k - 1$ . This can be seen by taking a graph where every subgraph has minimum degree at most  $k - 2$  and iteratively coloring the vertex with minimum degree and then removing it from the graph.

Therefore, the function  $h(k) = g\left(\frac{k-1}{2}\right)$  satisfies the desired properties.  $\square$

Note that Fact (ii) holds for the coloring number (the maximum of  $\delta(H) + 1$  over all subgraphs  $H$  of  $G$ ) as well, so Theorem 1.1 can be strengthened to show that  $ch_a$  grows with the coloring number.

To prove Theorem 2.1, consider any bipartite graph  $H$  with bipartition  $(A, B)$  where  $|A| \geq |B|$ . We will consider lists of size  $s$  taken from a color set of size  $s^5$ . We will show that there is a function  $f(s)$  such that if  $\delta(H) \geq f(s)$ , then there is an assignment of lists to vertices and labels to edges such that there is no proper adaptable coloring from these lists. This is sufficient to show that  $ch_a(H) > s$ . This clearly is sufficient to prove Theorem 2.1, as we can let  $g = f^{-1}$ .

We start with a few helpful definitions. An assignment of lists to  $A$  (resp.  $B$ ) is called an  $A$ -set (resp.  $B$ -set). Given a  $B$ -set, we say that  $a \in A$  is *supersurrounded* (inspired by “surrounded” from [1]) if every possible list of  $s$  elements from  $[s^5]$  appears in more than  $s^3$  lists on vertices in  $N(a)$  (the neighborhood of  $a$ ). Furthermore, we call the  $B$ -set *bad* if at least half of the vertices in  $A$  are supersurrounded.

Theorem 2.1 follows directly from the following two lemmas.

**Lemma 2.2.** *If  $\delta(H) \geq d = 36s^5 \binom{s^5}{s}$ , then there is a bad  $B$ -set of lists.*

**Lemma 2.3.** *There is an  $s_0$  such that for any bad  $B$ -set  $\mathcal{B}$ , if  $s \geq s_0$ , there is an assignment of colors to the edges of  $H$  and an  $A$ -set  $\mathcal{A}$  such that  $H$  does not have an acceptable coloring.*

*Proof of Theorem 2.1.* Let  $g$  be the inverse of the function  $f(s) = 36s^5 \binom{s^5}{s}$ . We choose a bad  $B$ -set  $\mathcal{B}$  according to Lemma 2.2. We choose an  $A$ -set  $\mathcal{A}$  and an edge coloring according to Lemma 2.3 such that there is no acceptable coloring from the assigned lists.  $\square$

*Proof of Lemma 2.2.* Uniformly at random assign lists to each of the vertices in  $B$ .

Let  $a \in A$  be an arbitrary vertex and let  $Y$  be the number of lists which do not appear more than  $s^3$  times in  $a$ 's neighborhood. We will show that the probability that  $a$  is not supersurrounded, i.e. that  $Y \geq 1$ , is less than  $1/2$ .

To make this computation it will be helpful to consider a single list. Let  $S \subseteq [s^5]$  an arbitrary list of size  $s$  and let  $X$  be the number of neighbors of  $a$  whose assigned list is  $S$ .

Since the lists are assigned uniformly at random, for each neighbor  $b$  of  $a$ , the probability that  $b$  is assigned  $S$  is  $1/\binom{s^5}{s}$ . Therefore:

$$\mathbb{E}(X) = \frac{|N(a)|}{\binom{s^5}{s}} \geq \frac{d}{\binom{s^5}{s}} = 36s^5$$

The Chernoff bound yields the following.

$$\begin{aligned} \Pr(X \leq s^3) &\leq \Pr\left(X \leq \frac{\mathbb{E}(X)}{2}\right) \leq \Pr\left(|X - \mathbb{E}(X)| > \frac{\mathbb{E}(X)}{2}\right) \\ &< 2e^{-[\mathbb{E}(X)/2]^2/[3\mathbb{E}(X)]} \\ &= 2e^{-\mathbb{E}(X)/12} \leq 2e^{-36s^5/12} = 2e^{-3s^5} \end{aligned}$$

Now we can bound the expected value of  $Y$  using the linearity of expectation.

$$\mathbb{E}(Y) = \binom{s^5}{s} \Pr(X \leq s^3) < \binom{s^5}{s} 2e^{-3s^5} \leq 2e^{s^5} e^{-3s^5} < \frac{1}{2}, \text{ for every } s \geq 1.$$

Markov's Inequality yields that the probability that  $a$  is not supersurrounded is:

$$\Pr(Y \geq 1) \leq E(Y) < \frac{1}{2}$$

Now let  $Z$  be the number of vertices in  $A$  which are supersurrounded. By the linearity of expectation,  $\mathbb{E}(Z) > \frac{1}{2}|A|$ . Thus the probability that  $Z \geq \frac{1}{2}|A|$  is positive, and therefore there is a bad  $B$ -set.  $\square$

*Proof of Lemma 2.3.* Assume that  $\mathcal{B}$  is a bad  $B$ -set.

**Step 1:** For each edge  $e = ab$  where  $a \in A$  and  $b \in B$ , assign to  $e$  a color uniformly at random from  $L(b)$ .

Consider any  $a \in A$  that is supersurrounded. Fix a coloring of  $B$  from the lists of  $\mathcal{B}$ .

We will say that a color  $c$  is *available* for  $a$  if there is no neighbor  $b$  of  $a$  such that  $ab$  is labeled  $c$  and  $b$  is colored  $c$ . A coloring of  $B$  is *extendable* to  $A$  if every vertex in  $A$  has at least one available color in its list. Note that  $G$  is colorable if and only if at least one coloring of  $B$  is extendable to  $A$ .

First we note that all but at most  $s - 1$  colors appear more than  $s^2$  times on vertices in the neighborhood of  $a$ . We can see this by assuming that  $c_1, \dots, c_s$  all appear at most  $s^2$  times in  $N(a)$ . So the list  $\{c_1, \dots, c_s\}$  can only appear in  $N(a)$  at most  $s \cdot s^2 = s^3$  times. However, as  $a$  is supersurrounded, the list appears more than  $s^3$  times and thus we have a contradiction.

Let  $c$  be a color that appears more than  $s^2$  times in  $N(a)$ . The probability that a color  $c$  is available for  $a$  is the probability that for every neighbor  $b$  of  $a$  such that  $b$  is colored  $c$ , the edge  $e = ab$  is not labeled  $c$ . Note that since we are choosing the color for  $e$  from  $b$ 's list of colors, the probability that  $e$  is colored the same as  $b$  is  $1/s$ . Therefore:

$$\Pr(c \text{ is available}) < \left(1 - \frac{1}{s}\right)^{s^2} < e^{-s}.$$

Define  $Z$  to be the number of available colors beyond the  $s - 1$  colors which may appear  $s^2$  or fewer times,

$$\mathbb{E}(Z) < s^5 e^{-s}$$

Using Markov's Inequality:

$$\Pr(Z \geq 1) \leq \mathbb{E}(Z) < s^5 e^{-s}.$$

Now, including the  $s - 1$  colors which may appear  $s^2$  or fewer times, we can with high probability bound the number of available colors as follows.

$$\Pr(\# \text{ available colors for } a \geq s) < s^5 e^{-s}. \tag{1}$$

**Step 2:** For each vertex  $a \in A$ , uniformly at random choose one of the  $\binom{s^5}{s}$  possible lists.

Now, assuming that  $a$  is a vertex with fewer than  $s$  available colors, we can bound the probability that the list chosen for  $a$  has an available color. Since there are at most  $s - 1$  colors available for  $a$ , the probability that a random color  $c$  is available to  $a$  is at most  $(s - 1)/s^5$ .

$$\Pr(\text{list chosen for } a \text{ contains an available color}) \leq s \cdot \frac{s - 1}{s^5} < \frac{1}{s^3}$$

Therefore, by (1), the probability that  $a$  has  $s$  or more available colors or the list chosen for  $a$  has an available color is less than  $s^5 e^{-s} + s^{-3}$ . For sufficiently large  $s$ , this is less than  $1/s^2$ .

Since  $\mathcal{B}$  is a bad  $B$ -set, there are at least  $\frac{1}{2}|A|$  supersurrounded vertices. Thus, remembering that  $|A| \geq |B|$ , we can bound the probability that every supersurrounded vertex has an available color in its list as follows.

$$\Pr\left(\begin{array}{l} \text{every supersurrounded vertex} \\ \text{has an available color in its list} \end{array}\right) < \left(\frac{1}{s^2}\right)^{\frac{1}{2}|A|} = s^{-|A|} \leq s^{-|B|}$$

Let  $W$  be the number of colorings of  $B$  which are extendable to  $A$ . Given a  $B$ -set  $\mathcal{B}$ , there are  $s^{|B|}$  possible ways of choosing colors for the vertices in  $B$ . Thus we can bound the expected value of  $W$  as follows.

$$\mathbb{E}(W) < s^{|B|} \cdot s^{-|B|} = 1$$

Since the expected value is less than 1, there must be a choice of an  $A$ -set and edge colorings such that no coloring of  $B$  can be extended to a coloring of  $A$ . □

- [1] N. Alon, Restricted colorings of graphs, *Surveys in Combinatorics, Proc. 14th British Combinatorial Conference, London Mathematical Society Lecture Notes Series* 187, edited by K. Walker, Cambridge University Press, 1993, pp. 1-33.
- [2] A. Archer, On the upper chromatic numbers of the reals, *Discrete Math*, 214:65-75, 2000.
- [3] H. Chernoff, A measure of the asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Annals of Mathematical Statistics*, 23:493-509, 1952.

- [4] P. Erdős and A. Gyárfás, Split and balanced colorings of complete graphs, *Discrete Math*, 200 (1999), 79-86.
- [5] J. Esperet, M. Montassier and X. Zhu, Adapted list colouring of planar graphs, submitted.
- [6] T. Feder and P. Hell, Full constraint satisfaction problems, *SIAM J. Computing*, 36 (2006), 230-246.
- [7] T. Feder, P. Hell, S. Klein and R. Motwani, Complexity of list partitions, *SIAM J. Discrete Mathematics*, 16 (2003), 449-478.
- [8] T. Feder, P. Hell and W. Xie, Matrix partitions with finitely many obstructions, *Electronic Notes in Discrete Math*, 28 (2007), 371-378.
- [9] A. Guan and X. Zhu, Adaptive choosability of planar graphs with sparse short cycles, *Discrete Math.*, 309 (2009), 6044-6047.
- [10] P. Hell, Z. Pan, T. Wong and X. Zhu, Adaptable colouring of graph products, preprint.
- [11] P. Hell and X. Zhu, Adaptable chromatic number of graphs, *European J. Combinatorics*, 29(4) (2008), 912-921.
- [12] A. Kostochka and X. Zhu, Adapted list coloring of graphs and hypergraphs, *SIAM J. Discrete Math*, 22 (2008), no. 1, 398-408.
- [13] M. Molloy and B. Reed, *Graph colouring and the probabilistic method*, Springer-Verlag, Berlin, 2001.
- [14] M. Molloy and G. Thron, An asymptotically tight bound on the adaptable chromatic number, *J. Graph Theory*, to appear.
- [15] M. Montassier, A. Raspaud and X. Zhu, An upper bound on adaptable choosability of graphs, *European Journal of Combinatorics*, 30 (2009), 351-355.