Starting At A Non-Zero Base

For which \( n \in \mathbb{N} \) is \( n^2 \leq 2^n \)?

<table>
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<th>( n )</th>
<th>( n^2 )</th>
<th>( 2^n )</th>
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<tr>
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It seems to be true for all \( n \neq 3 \). For \( n < 3 \) we can simply do the calculations (as we’ve done), but not for the infinite number of \( n \)’s \( \geq 4 \). Can inductive thinking handle this?

To get precise: for \( n \in \mathbb{N} \), let \( P(n) \) be:

\[ n^2 \leq 2^n \]

What if we prove instead:

\[ P(4) \land (P(4) \rightarrow P(5)) \land (P(5) \rightarrow P(6)) \land \cdots \]

In other words:

\[ P(4) \land \forall n \in \mathbb{N}, n \geq 4 \rightarrow (P(n) \rightarrow P(n + 1)) \]

Yes, inductive thinking can handle \( n \)’s \( \geq 4 \) (without having to do \( P(3) \) which in this case is false). We believe the following inductive principle:

\[ [P(4) \land \forall n \in \mathbb{N}, n \geq 4 \rightarrow (P(n) \rightarrow P(n + 1))] \rightarrow [\forall n \in \mathbb{N}, n \geq 4 \rightarrow P(n)] \]

Now the creative part: connecting \( n^2 \leq 2^n \) to \( (n + 1)^2 \leq 2^{n+1} \). We can first try something simpler: define \( a_n = n^2 \) and \( b_n = 2^n \) inductively, in other words connect \( n^2 \) to \( (n + 1)^2 \) and \( 2^n \) to \( 2^{n+1} \), in other words think about just the change in the left side and right side values (not the relation between the sides). The most obvious inductive definitions are:

\[ a_0 = 0 \]
\[ a_{n+1} = a_n + 2n + 1, \text{ for } n \in \mathbb{N} \]
\[ b_0 = 1 \]
\[ b_{n+1} = 2 \cdot b_n, \text{ for } n \in \mathbb{N} \]

One change is additive, the other multiplicative, so it may be hard to compare them. But \( b_n \) is not hard to give additively:

\[ b_0 = 1 \]
\[ b_{n+1} = b_n + b_n, \text{ for } n \in \mathbb{N} \]

so maybe we can think additively. Is adding \( 2n + 1 \) not more than adding \( b_n = 2^n \)? If so, then that gets us from the Inductive Hypothesis \( n^2 \leq 2^n \), to \( (n + 1)^2 \leq 2^{n+1} \). Unfortunately, it’s still a comparison of a polynomial and an exponential, so still hard to do directly. We could try to prove \( 2n + 1 \leq 2^n \) inductively, as a separate proof, hoping that it has an even simpler inductive step that we can do directly (it does: consider this an Exercise).

Instead, we’ll use the IH twice:

1. Once so that we only have to reason about the changes \( 2n + 1 \) and \( 2^n \). This is how induction helps in general.
2. A second time to reason about the change itself.
For (2): we’ll know that \( n^2 \leq 2^n \) so if we can show that \( 2n + 1 \leq n^2 \) then we know that \( 2n + 1 \leq 2^n \). Proving \( 2n + 1 \leq n^2 \) is a simple case of the \( O \) algebra of CSC165: 
\[
2n + 1 \leq 2n + n, \text{ for } n \geq 1 \\
= 3n \\
\leq n^2, \text{ for } n \geq 3
\]
Since we’ll be using this for \( n \geq 4 \), we’re okay. Now we’re ready for a careful proof.

Proof. By Induction.

Base Case: \( 4 \cdot 4^2 = 16 \leq 16 = 2^4 \).

Inductive Step Let \( n \in \mathbb{N} \), with \( n \geq 4 \). Suppose \( n^2 \leq 2^n \) (IH).
Then
\[
(n + 1)^2 = n^2 + 2n + 1 \\
\leq 2^n + 2n + 1 \text{ by (IH)} \\
\leq 2^n + 2n + n, \text{ since } n \geq 4 \geq 1 \\
= 2^n + 3n \\
\leq 2^n + n^2, \text{ since } n \geq 4 \geq 3 \\
\leq 2^n + 2^n \text{ by (IH)} \\
= 2 \cdot 2^n \\
= 2^{n+1}.
\]

\[\square\]

Exercise. A multiplicative inductive definition of \( a_n \) can be forced:
\[
a_{n+1} = (n + 1)^2 = \frac{(n + 1)^2}{n^2} \cdot n^2 = \left(1 + \frac{1}{n}\right)^2 n^2 = \left(1 + \frac{1}{n}\right)^2 a_n.
\]

For \( n \geq 4 \):
\[
(1 + \frac{1}{n})^2 \leq (1 + \frac{1}{4})^2 = \frac{25}{16} \leq \frac{32}{16} = 2, \text{ so } a_{n+1} \leq 2a_n. \text{ Use this reasoning for a different proof by induction.}
\]

Reaching Farther Back

What postage can be made from 4 and 7 cent stamps? In lecture we made a table, and it seemed that all postage of \( n \geq 18 \) could be made (and some smaller ones, but not 17). So for the infinite number of \( n \)'s \( \geq 18 \) we consider using induction.

To practice, consider two questions:

1. If we know we can make 237 (chosen instead of the 'obvious' multiple of 4:236), which larger values does this help us with?
2. If we want to make 237, which smaller values could help us?

In the first two weeks we’ve been thinking about \( P(n) \rightarrow P(n+1) \) or \( P(n-1) \rightarrow P(n) \). Now we’re thinking more generally about \( P(n_1) \rightarrow P(n_2) \) for \( n_1 < n_2 \). What hasn’t changed is that there is a \( P \) hypothesis and a \( P \) conclusion, so we can practice by thinking about \( P(237) \rightarrow \) and \( \rightarrow P(237) \). Some things to notice:

- Postage for 237 leads to postage for 237 + 4 by adding a 4 cent stamp. Similarly, postage for 233 leads to postage for 237.
- Something similar with 7 instead of 4.
- Once we have 237 = 59·4 + 1·7, we can get 238 by adding two 4 cent stamps and removing the one 7 cent stamp: 238 = 61·4 + 0·7. Now there are no 7 cent stamps to remove, so we can’t repeat what we just did. Instead we can add three 7 cent stamps and remove five 4 cent stamps.

This third idea can be made to work: Exercise.

Instead, we used the first idea, and noticed that if we manage postage for four \( n \)'s in a row, then we can get the next four \( n \)'s, the next four after those, etc. We made postage for 18, 19, 20 and 21, and then were convinced by the following Inductive principle:
\[
[P(18) \land P(19) \land P(20) \land P(21) \land \forall n \in \mathbb{N}, (n \geq 22 \land P(n-4)) \rightarrow P(n)] \rightarrow [\forall n \in \mathbb{N}, n \geq 18 \rightarrow P(n)].
\]
In other words, we believe the following recursive structure is valid:

```python
# For natural number n >= 18, return (a,b) such that n = 4a + 7b.
def postage(n):
    if (n == 18): return (1,2)
    elif (n == 19): return (3,1)
    elif (n == 20): return (5,0)
    elif (n == 21): return (0,3)
    else: # n >= 22
        (a0,b0) = postage(n-4) # n-4 >= 18
        return (a0+1,b0)
```

For \( n \in \mathbb{N} \), let \( P(n) \) be \( \exists a, b \in \mathbb{N}, n = 4a + 7b \). The implementation contains the details of the inductive proof.

**Proof.** By Induction.

**Base Cases:** 18, 19, 20, 21

\[
\begin{align*}
18 &= 4 \cdot 1 + 7 \cdot 2 \\
19 &= 4 \cdot 3 + 7 \cdot 1 \\
20 &= 4 \cdot 5 + 7 \cdot 0 \\
21 &= 4 \cdot 0 + 7 \cdot 3
\end{align*}
\]

**Inductive Step.** Let \( n \in \mathbb{N} \), with \( n \geq 22 \). Suppose \( n - 4 = 4a_0 + 7b_0 \) for some \( a_0, b_0 \in \mathbb{N} \) (IH). Then

\[
\begin{align*}
n &= (n - 4) + 4 \\
&= 4a_0 + 7b_0 + 4 \text{ by (IH)} \\
&= 4(a_0 + 1) + 7b_0
\end{align*}
\]

where \( a + 1, b \in \mathbb{N} \) since \( a, b \in \mathbb{N} \).

\( \square \)