(1) We prove that \( \forall m \in \mathbb{N}, \forall n \in \mathbb{N}, (1 + mn) \leq (1 + m)^n \).

**Proof.** Let \( m \in \mathbb{N} \).

Now by Simple Induction we prove \( \forall n \in \mathbb{N}, (1 + mn) \leq (1 + m)^n \).

**Base Case:** \( 0. \) \( (1 + m \cdot 0) = 1 \leq 1 = (1 + m)^0 \).

**Inductive Step** Let \( n \in \mathbb{N} \).

(IH) Assume \( (1 + mn) \leq (1 + m)^n \).

Then

\[
(1 + m)^{n+1} = (1 + m)^n \cdot (1 + m) \geq (1 + mn) (1 + m), \text{ by (IH),}
\]

\[
= 1 + mn + m + m^2 n = (1 + m (n + 1)) + mn^2
\]

\[
\geq (1 + m (n + 1)) \text{ (since } mn^2 \geq 0, \text{ since } m \geq 0).\]

(2) We prove that \( r_n \leq 236 \cdot \log_2 (\log_2 n) \) for all natural numbers \( n \geq 4 \).

**Proof.** By Complete Induction.

Let \( n \) be a natural number with \( n \geq 4 \).

**Base Cases** \( 4 \leq n \leq 15 \).

Then \( 1 = \left\lfloor \sqrt{\sqrt{4}} \right\rfloor \leq \left\lfloor \sqrt{\sqrt{n}} \right\rfloor \leq \left\lfloor \sqrt{\lfloor 16 \rfloor} \right\rfloor = 1, \) so \( \left\lfloor \sqrt{\sqrt{n}} \right\rfloor \) = 1.

So

\[
r_n = 1 + r_{\lfloor \sqrt{n} \rfloor} = 1 + \left( 1 + r_{\lfloor \sqrt{\lfloor \sqrt{n} \rfloor} \rfloor} \right) = 2 + r_1 = 3
\]

\[
\leq 236 \cdot 1 = 236 \log_2 2 = 236 \log_2 (\log_2 4) \leq 236 \log_2 (\log_2 n),
\]

since \( 4 \leq n \) and \( \log_2 \circ \log_2 \) is increasing.

**Inductive Step** Let \( n \in \mathbb{N} \) with \( 16 \leq n \).

(IH) Suppose \( r_k \leq 236 \log_2 (\log_2 k) \) for each \( k \in \mathbb{N} \) such that \( 4 \leq k < n \).

Since \( 16 \leq n \): \( 4 = \lfloor \sqrt{16} \rfloor \leq \lfloor \sqrt{n} \rfloor \leq \sqrt{n} < n \), so the (IH) applies for \( k = \lfloor \sqrt{n} \rfloor \).

Then

\[
r_n = 1 + r_{\lfloor \sqrt{n} \rfloor}
\]

\[
\leq 1 + 236 \log_2 (\log_2 \lfloor \sqrt{n} \rfloor), \text{ from (IH) as noted above,}
\]

\[
\leq 1 + 236 \log_2 (\log_2 \sqrt{n}), \text{ since } \log_2 \circ \log_2 \text{ is increasing and } \lfloor \sqrt{n} \rfloor \leq \sqrt{n},
\]

\[
= 1 + 236 \log_2 \left( \frac{1}{2} \log_2 n \right)
\]

\[
= 1 + 236 (-1 + \log_2 (\log_2 n))
\]

\[
= (1 - 236) + 236 \log_2 (\log_2 n) \leq 236 \log_2 (\log_2 n).
\]
(3) (a) Define $b$ by:

\[
\begin{align*}
b_0 &= 1, \\
b_h &= 2b_{h-1} (b_0 + \cdots + b_{h-1}) - b_{h-1}^2, \ h \geq 1.
\end{align*}
\]

Claim: for all natural numbers $h$, $b_h$ is the number of binary trees of height $h$.

**Proof.** By Complete Induction.

**Base Case:**

There is exactly one empty tree, and $b_0 = 1$, so $b_0$ is the number of binary trees of height 0.

**Inductive Step**

Let $h \in \mathbb{N}$ with $1 \leq h$.

(IH) Suppose $b_i$ is the number of binary trees of height $i$, for each $i \in \mathbb{N}$ such that $0 \leq i < h$. A binary tree of height $h \geq 1$ is determined by its left and right subtrees, which are binary trees of height less than $h$, with one of them having height exactly $h - 1$.

A tree of height less than $h$ has height 0, 1, \ldots, or $h - 1$, and the number of trees of each of those heights is $b_0, b_1, \ldots, b_{h-1}$ (by the (IH) for $i = 0, 1, \ldots, h - 1 < h$).

So the number of trees of height less than $h$ is $b_0 + \cdots + b_{h-1}$.

If the left subtree has height $h - 1$ there are (by (IH)) $b_{h-1}$ possibilities, multiplied by the $b_0 + \cdots + b_{h-1}$ possibilities for the right subtree. There are the same amount again if we switch left and right, doubling the total. That double-counts the case where the left and right subtrees both are of height $h - 1$, so subtract off the number of those: $b_{h-1} \cdot b_{h-1} = b_{h-1}^2$.

(b) Claim: $b_{h+1} = a_{h+1}^2 - a_h^2$ for all natural numbers $h$.

**Proof.** By Complete Induction.

**Base Case:**

\[
b_{0+1} = b_1 = 2b_0 (b_0) - b_0^2 = 2 - 1 = 1 = (0^2 + 1)^2 - 0^2 = (a_0^2 + 1)^2 - a_0^2 = a_{0+1}^2 - a_0^2.
\]

**Inductive Step**

Let $h \in \mathbb{N}$ with $1 \leq h$. Note that $h - 1 \in \mathbb{N}$, which we'll use a few times.

(IH) Suppose $b_{i+1} = a_{i+1}^2 - a_i^2$ for $i = 0, \ldots, h - 1$.

Then

\[
b_{h+1} = 2b_h (b_0 + \cdots + b_h) - b_h^2
\]

where splitting out $[b_1 + \cdots + b_h]$ is valid since $1 \leq h$. From (IH) for $i = 0, \ldots, h - 1$, we get

\[
= 2b_h (b_0 + [(a_1^2 - a_0^2) + (a_2^2 - a_1^2) + (a_3^2 - a_2^2) + \cdots + (a_h^2 - a_{h-1}^2)]) - b_h^2
\]

\[
= 2b_h (b_0 + a_h^2 - a_0^2) - b_h^2
\]

\[
= 2b_h (1 + a_h^2) - b_h^2
\]

\[
= b_h (2 + 2a_h^2 - b_h)
\]

\[
= (a_h^2 - a_{h-1}^2) (2 + 2a_h^2 - (a_h^2 - a_{h-1}^2)) \quad \text{(from (IH) for } i = h - 1)\]

\[
= (a_h^2 - a_{h-1}^2) (2 + a_h^2 + a_{h-1}^2)
\]

\[
= ((a_{h+1} - 1) - (a_h - 1)) (2 + (a_{h+1} - 1) + (a_h - 1)) \quad \text{(from } a_n \text{ for } n = h, h - 1 \in \mathbb{N})
\]

\[
= (a_{h+1} - a_h) (a_{h+1} + a_h)
\]

\[
= a_{h+1}^2 - a_h^2.
\]