Midterm Solutions

(1) <u>Base Cases: $n = 0, 1$</u>: $\left(x + \frac{1}{x}\right)^0 = 1 \in \mathbb{N}$. $\left(x + \frac{1}{x}\right)^1 \in \mathbb{N}$ is given.
<u>Inductive Step</u> Let $n \in \mathbb{N}$, $n \geq 1$. Suppose $x^k + \frac{1}{x^k} \in \mathbb{N}$ for all $k \leq n$.
Then using the IH with $k = 1$ and $k = n$, the following is in $\mathbb{N}$:

$$\left(x + \frac{1}{x}\right)\left(x^n + \frac{1}{x^n}\right) = x^{n+1} + \frac{1}{x^{n-1}} + x^{n-1} + \frac{1}{x^{n+1}} = \left(x^{n+1} + \frac{1}{x^{n+1}}\right) + \left(x^{n-1} + \frac{1}{x^{n-1}}\right).$$

So then, using the IH with $k = n - 1$, the following is in $\mathbb{N}$:

$$\left(x + \frac{1}{x}\right)\left(x^n + \frac{1}{x^n}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right) = x^{n+1} + \frac{1}{x^{n+1}}.$$

(2) <u>Base Cases: $n = 3, 4, 5$</u>:

$$f(3) = 23 \geq 9 = 3^2$$
$$f(4) = 236 \geq 16 = 4^2$$
$$f(5) = f(5-2) + 4f\left(\left\lfloor \frac{5}{2} \right\rfloor\right) - 3 = f(3) + 4f(2) - 3 = 23 + 4 \cdot 3 - 3 = 31 \geq 25 = 5^2.$$

<u>Inductive Step</u> Let $n \in \mathbb{N}$, $n \geq 6$. Suppose $f(k) \geq k^2$ for $3 \leq k < n$.
Then $3 \leq 4 \leq n - 2 < n$ and $3 = \left\lfloor \frac{6}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} < n$, so applying the IH to $n - 2$ and $\left\lfloor \frac{n}{2} \right\rfloor$:

$$\begin{aligned}
f(n) &= f(n-2) + 4f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - 3 \\
&\geq (n-2)^2 + 4\left(\left\lfloor \frac{n}{2} \right\rfloor\right)^2 - 3 \\
&\geq (n-2)^2 + 4\left(\frac{n-1}{2}\right)^2 - 3 \\
&= n^2 - 4n + 4 + n^2 - 2n + 1 - 3 \\
&= n^2 + \left(n^2 - 6n + 2\right) \\
&\geq n^2 + \left(n^2 - n \cdot n + 2\right), \text{ since } n \geq 6 \\
&\geq n^2.
\end{aligned}$$

(3) (a)

$$a_0 = 0.$$
$$a_1 = a_0 + (list[0] - a_0)/(0+1) = list[0].$$
$$a_2 = a_1 + (list[1] - a_1)/(1+1) = list[0] + (list[1] - list[0])/2 = (list[1] + list[0])/2.$$
$$a_3 = a_2 + (list[2] - a_2)/(2+1) = \frac{3a_2 + list[2] - a_2}{3} = \frac{list[2] + 2a_2}{3} = \frac{list[2] + list[1] + list[0]}{3}.$$

(b) $I(k)$ is: $i_k = k$ and if $k > 0$ then $a_k = \frac{list[k-1] + \cdots + list[0]}{k}$.

(c) We prove $I(k)$ for all $k \in \mathbb{N}$.
<u>Base Case $k = 0$</u>: $i_0 = 0$, and since $k \not> 0$, the rest is vacuously true.
<u>Inductive Step</u> Let $k \in \mathbb{N}$, $k \geq 1$. Suppose $I(k)$.
Then $i_{k+1} = i_k + 1 = k + 1$.
If $k = 1$ then from (a) we have $a_1 = list[0] = \frac{list[1-1]}{1}$.
If $k > 1$ then using $I(k)$:

$$\begin{aligned}
a_{k+1} = a_k + \frac{list[i_k] - a_k}{k+1} &= \frac{list[k] + (k+1)a_k - a_k}{k+1} = \frac{list[k] + ka_k}{k+1} \\
&= \frac{list[(k+1)-1] + list[k-1] + \cdots + list[0]}{k+1}.
\end{aligned}$$

Now, suppose the loop terminates at iteration k. Then by the loop condition, $i_k = n$ so $k = n$ so $I(n)$.
Thus , if $n > 0$ then $a_n = \frac{list[n-1] + \cdots + list[0]}{n}$. But $n > 0$ by the precondition, so we're done.