Option Pricing with Transaction Costs (Part1)

Outline:

- What a financial derivative, especially an option, is
- Hedging and Arbitrage
- What a Black-Scholes equation is and why we use it
- How we solve the Black-Scholes equation
- Leland's modification with transaction costs and our experiments

What is a financial derivative?

A financial derivative is a financial instrument, which derives its value from one or several underlying entities, it doesn't not have intrinsic value itself. It usually involves two parties.
The underlying entities could be stocks, index, interest rate and even weather!

What is an option?

An option is an important type of derivatives, it gives the holder a right to do something, but there is no obligation attached to it.

- Options
  - European Option
    - Put Option: Give the holder a right to sell a stock at price K at time T
    - Call Option: Give the holder a right to buy a stock at price K at time T
  - American Option
    - Similar to European Option, but the holder can exercise the option anytime before time T
  - Asian and Exotic Option
    - Similar idea, but much more complicated and lucrative

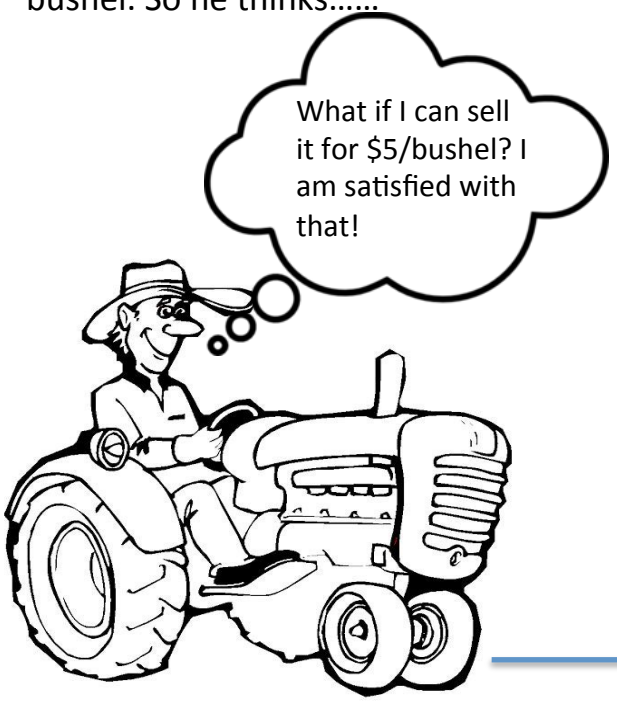
K: Strike Price
T: Maturity Date

Hedging, speculation and Arbitrage

Hedging is intended to reduce any substantial gain and loss in a business.
Arbitrage is to make money without any investment without risk.

Example :
hedging and speculation

It's August right now. George is a farmer, he estimates that he can harvest 5000 bushels of corns in September. He will need $20000 dollars for living and the seeds next year. So he at least have to make $4/bushel. So he thinks……



So he found Peter and paid him $500 in return that he will have the right to sell 5000 bushels at a price of $5 dollars/bushel in September to Peter.

So What can Happen?

2 cases:

1

If the corn price falls to $3/bushel in September, George will certainly find Peter and say:
Hi Peter! We have an agreement that you have to buy my corn for $5/bushel if I want to!

In this case: If George didn't hedge using the option, he will have
$3 x 5000=$15000
If he hedged he will have
$5 x 5000-$500=$24500

George successfully decreased his loss in this case !

2

If the corn price rises to $6/bushel in September, George will certainly NOT find Peter but sell his corn at the market price.

In this case: If George didn't hedge using the option, he will have
$6 x 5000=$30000
If he hedged he will have
$30000-$500=$29500

George decreased his gain in this case !

Peter has some insider information, and he knows the corn price at September will likely to be above $5/bushel, so George will not exercise the option. So he is willing to take the risk!
However, he also has a chance to loose a lot of money, if the corn price goes down.

Behavior like Peter's is called speculation !

Example :
Arbitrage

Unlike George, Peter also has other business, he constantly monitors the US dollar fixed interest rate between China and US.

If he finds out that the fixed cost for borrowing one dollar from China is 5 cents per year.

He also finds out that he can lend a dollar out for a fixed interest rate 6 cents per year in the US.

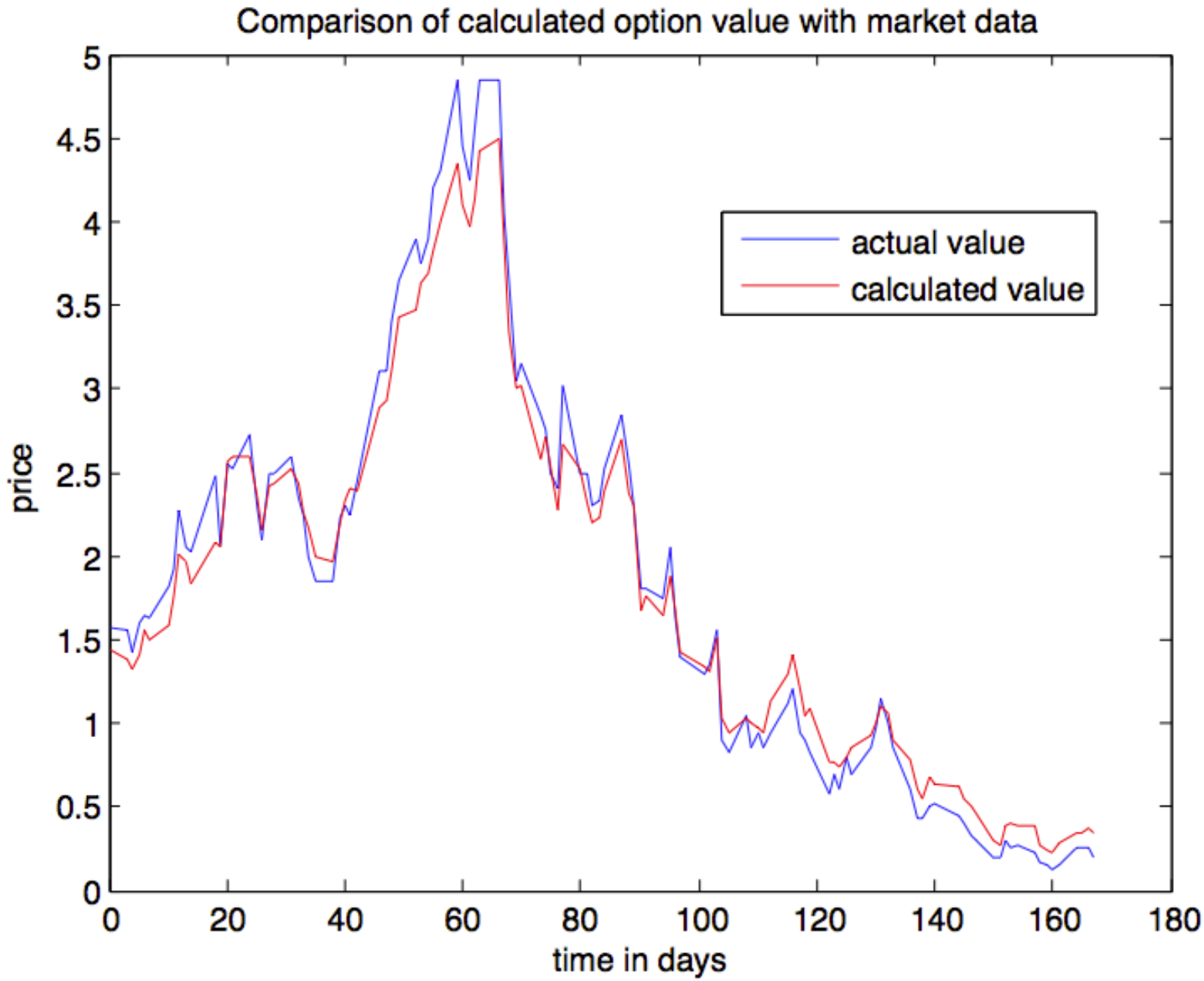
So what will he do?

If there are thousands of people like Peter, what will happen?

Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

V: Option Price, which is a function of S and t, unknown function
S: Underlying Stock Price, independent variable
t: Time to Maturity, independent variable
r: Risk-free Interest Rate, given constant
σ: Volatility, given constant


Comparison of calculated option value with market data
actual value
calculated value
price
time in days
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
20
40
60
80
100
120
140
160
180

Why is Black-Scholes equation important?

Black-Scholes equation is based on a lot of assumptions, which will be introduced later. However, if all the assumptions are fulfilled

At any time and no matter what the stock price is, an option price must satisfy the Black-Scholes equation, otherwise, there would be an arbitrage opportunity.

How do we solve Black-Scholes equation?

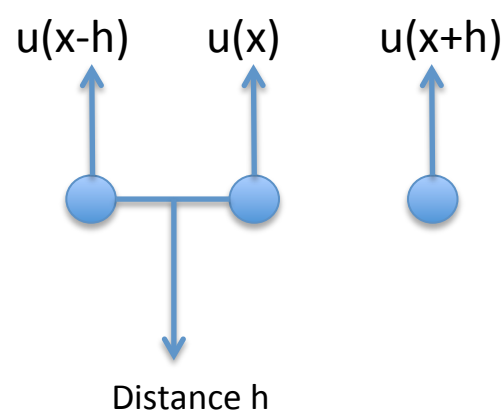
We use a second order finite difference method as our basic to solve Black-Scholes equation. We also incorporated more advanced numerical algorithms such as variable time step and non-uniform space grid.

A finite difference is a way to approximate derivative values by linear combinations of nearby function values:

Finite Difference Example:

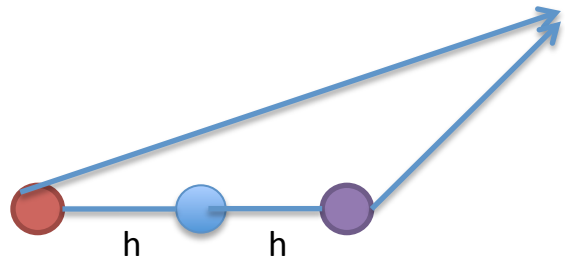
$$u'(x) \approx \frac{u(x+h) - u(x-h)}{2h}$$

$$u''(x) \approx \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$



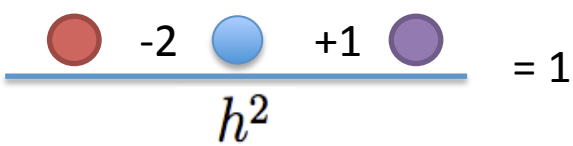
If we want to solve:

$$u''(x) = 1$$



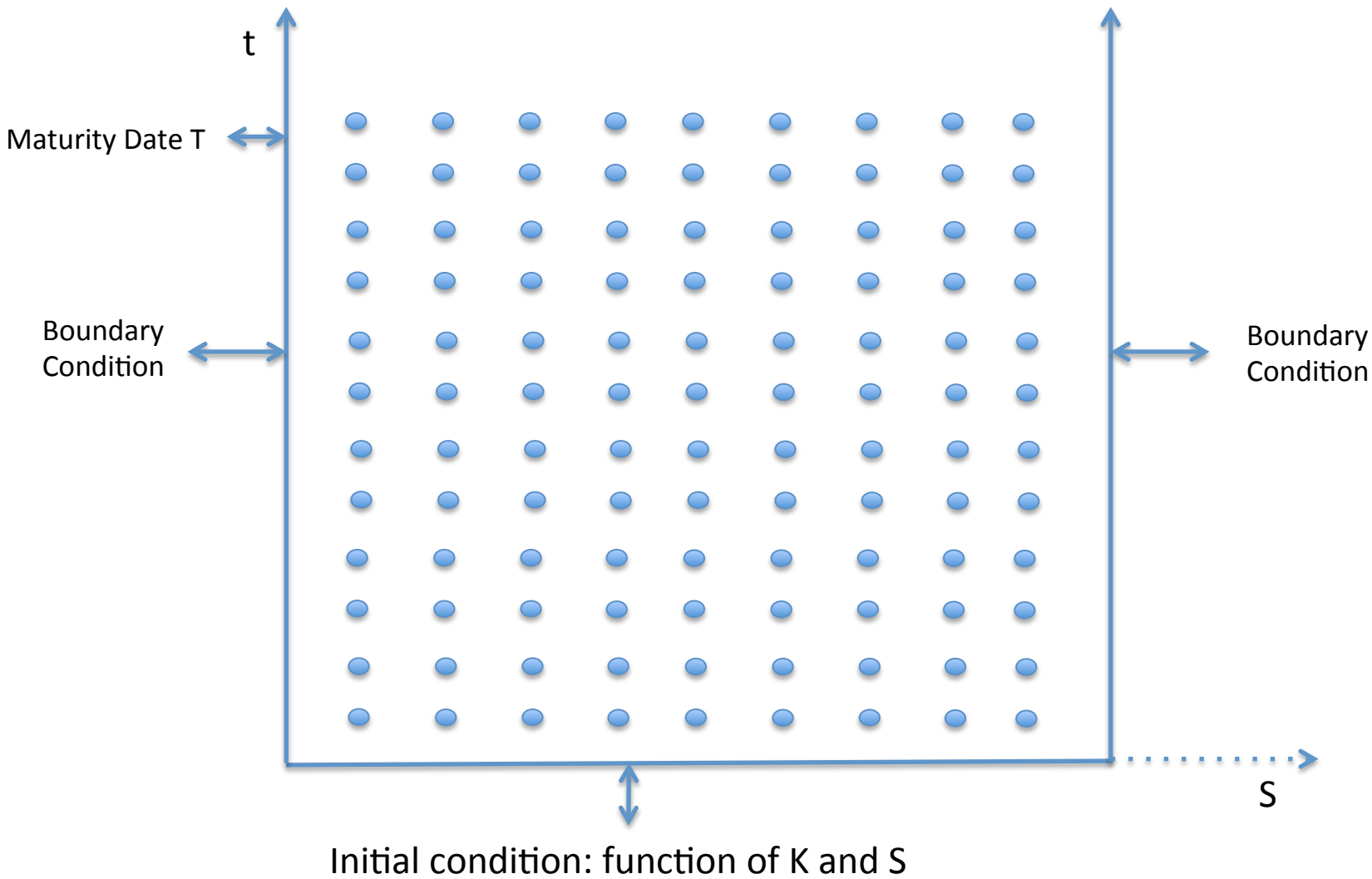
Blue and Purple are on the boundary, so we call them boundary conditions. Fortunately, we know the boundary condition for Black-Scholes equation

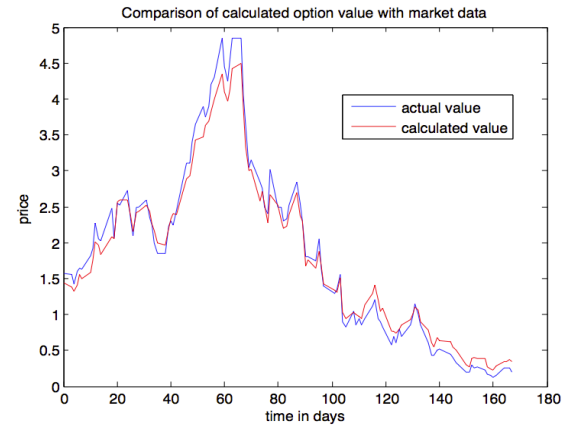
$$u''(x) \approx \frac{u(x+h) - 2u(x) + u(x+h)}{h^2}$$



= 1

Since we know the boundary condition, this is one equation and one unknown, therefore, we can easily solve the value of the blue bubble


t
Maturity Date T
Boundary Condition
Boundary Condition
S
Initial condition: function of K and S



Does every option on the market have the Black-Scholes value?

No! Because the assumption of Black-Scholes equation is hard to meet.

6 assumptions of Black-Scholes equation

1. Asset prices are lognormally distributed
2. The volatility and interest rate are constant through the life of an option
3. It's always possible to buy and sell any amount, even fractional, of stock in that constant interest rate
4. The underlying security does not pay a dividend
5. Buying and Selling do not incur any transaction cost
6. There is no tax involved whatsoever

Our research focuses on solving this problem.

Based on the suggestions by Leland (1985), Option prices with transaction cost satisfy a modified Black-Scholes equation with a different volatility

Recall Black-Scholes equation:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

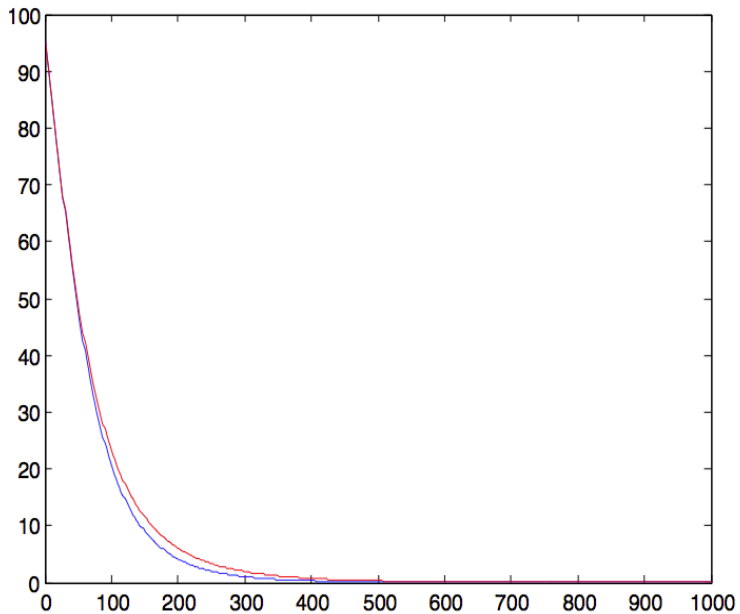
Modification

If we have a long position in option, i.e. we own (hold) an option, we have:

$$\sigma^2 = \sigma^2 - 2\kappa\sigma\sqrt{\frac{2}{\pi\Delta t}}$$

If we have a short position in option, i.e. we owe (write) an option, we have:

$$\sigma^2 = \sigma^2 + 2\kappa\sigma\sqrt{\frac{2}{\pi\Delta t}}$$



Sample graph of Option Price vs S without TC (blue), with TC (red) if we have a short position in option

Research Challenge:

Is this model reasonably correct?

If it is reasonably correct, how to choose κ?

How to choose Δt ?