Duration: **50 minutes**
Aids Allowed: **none**

Student Number: |_|_|_|_|_|_|_|_|_|_|_|

Family Name(s): _____

Given Name(s): _____

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*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This term test consists of 3 questions on 10 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, write your student number where indicated at the bottom of every odd-numbered page (except page 1), and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result or theorem covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

If you are unable to answer a question (or part), you will get 20% of the marks for that question (or part) if you write "I don't know" and nothing else—you will *not* get those marks if your answer is completely blank, or if it contains contradictory statements (such as "I don't know" followed or preceded by parts of a solution that have not been crossed off).

MARKING GUIDE

1: _____/10

2: _____/13

3: _____/13

BONUS MARKS: _____/ 5

TOTAL: _____/36

Good Luck!

OVER...

Use this page for rough work—clearly indicate any section(s) to be marked.

Question 1. [10 MARKS]

Consider the following algorithm (where "count" means "number of occurrences").

```
# Precondition: A is a list, b, e ∈ ℕ, 0 ⩽ b ⩽ e < len(A), A[b...e] are comparable with x
COUNT(x, A, b, e):
    if e == b:
        if A[b] == x:   return 1
        else:   return 0
    step = 1
    sum = 0
    c = b
    # LI: b ⩽ c ⩽ e + 1 and sum = count of x in A[b...c − 1]
    while c <= e:
        d = min(c + step − 1, e)
        sum = sum + COUNT(x, A, c, d)
        step = step * 2
        c = d + 1
    return sum
# Postcondition: returns the count of x in A[b...e]
```

Part (a) [8 MARKS]

Give an **exact** recurrence relation satisfied by $T(n)$, the worst-case running time of COUNT on inputs of size n, when n is a power of 2 (*i.e.*, $n = 2^k$ for some $k \in \mathbb{N}$)—your solution does **not** have to apply to other values of n. Justify that your recurrence is correct—in particular, specify clearly what you measure (*i.e.*, how you count steps) and give a precise definition of n in terms of the algorithm's parameters.

Part (b) [2 MARKS]

Does the Master Theorem apply to your recurrence relation from part (a)? Justify—simply explain why the theorem applies or not; do **not** try to give a closed-form expression for $T(n)$.

Use this page for rough work—clearly indicate any section(s) to be marked.

Question 2. [13 MARKS]

Write a detailed proof that the loop invariant in algorithm COUNT is correct. In your proof, you may simply **assume** that (x, A, b, e) is an input of size n that satisfies the precondition, and that COUNT is correct for all inputs of size *less* than n.

Use this page for rough work—clearly indicate any section(s) to be marked.

Question 3. [13 MARKS]

Write a detailed proof that algorithm COUNT is correct. In your proof, you may simply **assume** that the loop invariant is true and that the loop terminates—*i.e.*, you can answer this question even if you did not answer the previous one.

Use this page for rough work—clearly indicate any section(s) to be marked.

Bonus. [5 MARKS]

WARNING! This question is difficult and will be marked harshly: credit will be given only for making *significant* progress toward a correct answer (in particular, "I don't know" will be worth zero). Please attempt this only *after* you have completed the rest of the test.

Find and prove a tight bound on $T(n)$, the worst-case running time of COUNT on inputs of size n, based on your recurrence from the first question.

On this page, please write nothing except your name.

Family Name(s): _____

Given Name(s): _____