Duration: **50 minutes**
Aids Allowed: **none**

Student Number: |_|_|_|_|_|_|_|_|_|

Family Name(s): ______________________________

Given Name(s): ______________________________

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*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This term test consists of 3 questions on 10 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, write your student number where indicated at the bottom of every odd-numbered page (except page 1), and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result or theorem covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

If you are unable to answer a question (or part), you will get 20% of the marks for that question (or part) if you write "I don't know" and nothing else—you will *not* get those marks if your answer is completely blank, or if it contains contradictory statements (such as "I don't know" followed or preceded by parts of a solution that have not been crossed off).

MARKING GUIDE

# 1:	______/12
# 2:	______/12
# 3:	______/12
BONUS MARKS:	______/ 5
TOTAL:	______/36

 Good Luck!

Use this page for rough work—clearly indicate any section(s) to be marked.

Question 1. [12 MARKS]

Consider the sequence of numbers $a_0, a_1, a_2, \ldots$ defined as follows:

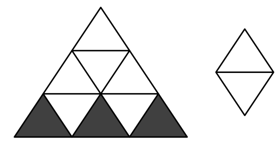
$$a_0 = 3, \qquad a_1 = 7, \qquad a_n = 3a_{n-1} - 2a_{n-2} \quad (\text{for } n \geqslant 2).$$

Write a detailed, carefully structured proof that $\forall\, n \in \mathbb{N}, a_n = 2^{n+2} - 1$. (Note: A significant number of marks will be assigned for having a clear and correct proof structure.)

Use this page for rough work—clearly indicate any section(s) to be marked.

Question 2. [12 MARKS]

A “sawtooth” grid of size n is a triangular grid with n triangles on a side and the bottom n triangles removed—as pictured on the right, for $n = 3$. A “diamond” tile consists of two triangles side-by-side—as pictured to the right of the grid. Use the *Principle of Well Ordering* (“P.W.O.”) to write a detailed, carefully structured proof that every sawtooth grid can be tiled with diamond tiles.



(Note: A significant number of marks will be assigned for having a clear and correct proof structure. *Proofs that do not make use of well ordering will receive at most half of the marks.*)

Use this page for rough work—clearly indicate any section(s) to be marked.

Question 3. [12 MARKS]

Consider the set T defined recursively as follows:

- $2 \in T$,
- if $x \in T$ and $x > 1$, then $x^2 \in T$,
- if $x \in T$ and $x > 1$, then $x/2 \in T$,
- T contains no other element.

Use *Structural Induction* to write a detailed, carefully structured proof that $\forall\, x \in T, \exists\, n \in \mathbb{N}, x = 2^n$.
(Note: A significant number of marks will be assigned for having a clear and correct proof structure. *Proofs that do not make use of structural induction will receive at most half of the marks.*)

Use this page for rough work—clearly indicate any section(s) to be marked.

Bonus. [5 MARKS]

WARNING! This question is difficult and will be marked harshly: credit will be given only for making *significant* progress toward a correct answer (in particular, "I don't know" will be worth zero). Please attempt this only *after* you have completed the rest of the test.

Consider the set T defined in Question 3. Write a detailed, carefully structured proof that $\forall\, n \in \mathbb{N}, 2^n \in T$.

On this page, please write nothing except your name.

Family Name(s): ______________________________

Given Name(s): ______________________________