

Worth: 8%

Due: By 6pm on Wednesday 30 September

For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks **will** be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.

1. Suppose $c \in \mathbb{R}^+$. In this question, you will investigate the existence of a value $d \in \mathbb{R}$ such that

$$\frac{(1 - a_1)(1 - a_2) \dots (1 - a_n)}{(1 + a_1)(1 + a_2) \dots (1 + a_n)} \geq d. \quad (1)$$

for all natural numbers $n \geq 1$ and all $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n a_i = c$.

- (a) Consider a specific value of c before tackling the general case: pick a value for c and make up appropriate values for each a_i for the cases $n = 1, 2, \dots, 5$. For example, if $c = 1$ and $n = 3$, then $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{3}$, and $a_3 = \frac{1}{6}$ would be appropriate values.

For your choice of c , what might be a good value for d ?

- (b) Prove that $\forall x, y \in \mathbb{R}^+$,

$$\frac{(1 - x)(1 - y)}{(1 + x)(1 + y)} \geq \frac{1 - x - y}{1 + x + y}.$$

You should use the proof method that results in the most elegant solution.

- (c) Based on your experiments and results from the previous parts, choose an appropriate value for d (i.e., the largest possible value) and prove Equation 1 for all natural numbers $n \geq 1$ and all $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n a_i = c$.

2. Prove that every rational number $0 < \frac{p}{q} < 1$ can be written as a finite sum $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_k}$, where $0 < a_1 < a_2 < \dots < a_k$ are all natural numbers.

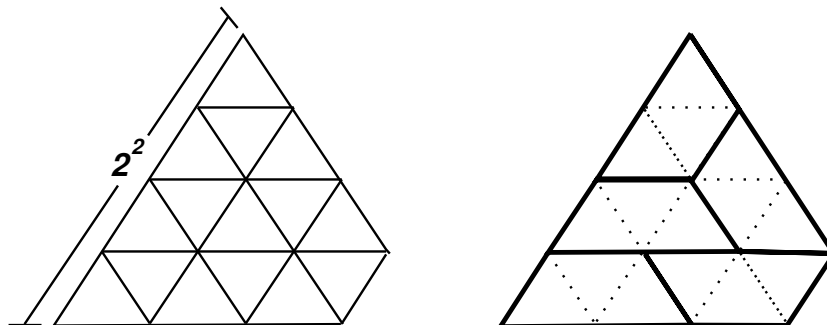
For example, $\frac{3}{10} = \frac{1}{4} + \frac{1}{20}$.

3. *Restricted Arithmetic Expressions* (“RAE’s” for short) are defined recursively as follows:

- $2, 3, 4, \dots$ are “atomic” RAE’s,
- if x, y are RAE’s, then so are $(x + y)$ and $(x \times y)$,
- nothing else is a RAE.

Prove that $\forall n \in \mathbb{N}$, every RAE that contains n atomic sub-expressions has value at least $2n$.

4. Use the Principle of Well Ordering to prove that $\forall n \in \mathbb{N}$, it is possible to tile a triangular grid with 2^n triangles on a side and one corner missing, using tiles that consist of three small triangles in a line. For example, for $n = 2$, the picture on the right shows one possible way to tile the grid on the left.



5. Consider the following description of a game.

- There are n people playing, one of whom leads the game. They are playing on a playing field with no obstacles. Everyone carries one water balloon.
- Everyone walks around the playing field until the game leader yells “stop”. Assume that the leader will yell stop only when everyone is in a position where they have a *unique* closest neighbour.
- Next, the leader yells “throw” and everyone throws their water balloon at their nearest neighbour (there is exactly one, by the assumption in the previous point).
- A *survivor* is anyone who is still dry at the end of the game (assuming everyone has perfect aim and the water balloons are designed so that they get only their intended target wet).

Prove that for all odd natural numbers n , there will always be at least one survivor when the game is played with n people.

(For your own amusement, what can you prove about the minimum and maximum number of survivors possible for arbitrary values of n —odd or even? What other properties can you prove? For example, think about how many balloons someone could be hit with, how many pairs of players could throw balloons at each other, *etc.*)