

Duration: **50 minutes**
 Aids Allowed: **NONE** (in particular, no calculator)

Student Number:

Last (Family) Name(s): _____

First (Given) Name(s): _____

*Do **not** turn this page until you have received the signal to start.*
 In the meantime, please read the instructions below carefully.

This term test consists of 3 questions on 10 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, write your student number where indicated at the bottom of every odd-numbered page (except page 1), and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result or theorem covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do — part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

If you are unable to answer a question (or part), you will get 20% of the marks for the question (or part) if you write “I don’t know” and nothing else — you will *not* get those marks if your answer is completely blank, or if it contains contradictory statements (such as “I don’t know” followed or preceded by parts of a solution that have not been crossed off).

MARKING GUIDE

1: _____/12

2: _____/ 9

3: _____/12

BONUS

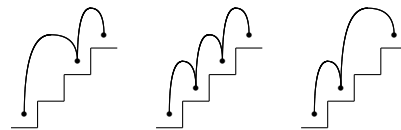
MARKS: _____/ 6

TOTAL: _____/33

Use this page for rough work — clearly indicate any section(s) to be marked.

Question 1. [12 MARKS]

Suppose that when you climb stairs, you always take the steps one or two at a time. Let C_n be the number of different ways that you can climb a staircase with n steps. For example, there are 3 different ways to climb a staircase with 3 steps, as illustrated on the right, so $C_3 = 3$. By convention, there is exactly 1 way to “climb” a staircase with 0 steps.



Part (a) [7 MARKS]

Give a recurrence relation for the values of C_n , including base cases, and briefly justify (explain).

Part (b) [5 MARKS]

Prove that $\forall n \in \mathbb{N}, \phi^{n-1} \leq C_n \leq \phi^n$, where $\phi = (1 + \sqrt{5})/2$ (remember that $\phi^2 = \phi + 1$).

Use this page for rough work — clearly indicate any section(s) to be marked.

Question 2. [9 MARKS]

Perform repeated substitution on the following recurrence to obtain an approximate closed form expression for $T(n)$. Show your work and simplify your expression as much as possible. There is no proof required for this question!

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1, \\ 2T(\lceil n/4 \rceil) + \sqrt{n} & \text{if } n > 1. \end{cases}$$

Use this page for rough work — clearly indicate any section(s) to be marked.

Question 3. [12 MARKS]

Write a detailed proof that the following algorithm is correct, following the format given in class.

```
# Precondition:  $x, y \in \mathbb{N}$ 
# Postcondition: returns  $x^y$ 
P( $x, y$ ):
1. if  $y == 0$ :
2.     return 1
3. elif  $y \% 2 == 0$ :    #  $y$  is even
4.      $t = P(x, y/2)$ 
5.     return  $t * t$ 
   else:
6.      $t = P(x, (y - 1)/2)$ 
7.     return  $t * t * x$ 
```

Use this page for rough work — clearly indicate any section(s) to be marked.

Bonus. [6 MARKS]

WARNING! This question is difficult and will be marked harshly: credit will only be given for making *significant* progress toward a correct answer — in particular, “I don’t know” will be worth zero (0). Please attempt this only *after* you have completed the rest of the test.

Prove a good *upper bound* for $T(n)$ from Question 2, *without using the Master Theorem*.

On this page, please write nothing except your name.

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