

NOTE TO STUDENTS: This file contains sample solutions to the term test together with the marking scheme and comments for each question. Please read the solutions and the marking schemes and comments carefully. Make sure that you understand why the solutions given here are correct, that you understand the mistakes that you made (if any), and that you understand *why* your mistakes were mistakes.

Remember that although you may not agree completely with the marking scheme given here it was followed the same way for all students. We will remark your test only if you clearly demonstrate that the marking scheme was not followed correctly.

For all remarking requests, please submit your request **in writing** directly to your instructor. For all other questions, please don't hesitate to ask your instructor during office hours or by e-mail.

Question 1. [10 MARKS]

Part (a) [3 MARKS]

State the Principle of Well Ordering.

SAMPLE SOLUTION:

Every non-empty set of natural numbers contains a smallest element.

MARKING SCHEME:

- **Correct statement:** [3 marks]

Must include each of the elements: “every set”, “of natural numbers”, “that is not empty”, “contains a smallest element”.

MARKER'S COMMENTS: There were four distinct items that needed to be in the statement (see marking scheme).

- You only lost 0.5 marks if you had essentially the right statement but forgot 1 element.
- More elements, or other errors and you lost more marks.
- Most people did well on this question.

Other errors that came up:

- Trying to write “the set S is non-empty” in symbolic form as “ $S \neq \{\emptyset\}$ ” isn't strictly correct: this is saying that S is not the set that contains only the empty set. What you really want to say is “ $S \neq \emptyset$ ”.

Part (b) [3 MARKS]

State the Principle of Complete Induction in symbolic form, for an arbitrary predicate $P(n)$.

SAMPLE SOLUTION:

$$(\forall n, (\forall k < n, P(k)) \rightarrow P(n)) \rightarrow \forall n, P(n)$$

MARKING SCHEME:

- **Notation:** [1 mark]
Correct use of symbolic notation for answer.
- **Correctness:** [2 marks]
Correct statement of complete induction.

MARKER'S COMMENTS:

- A number of people missed the latter part of the implication in the statement, *i.e.*, people would write $(\forall n, (\forall k < n, P(k)) \rightarrow P(n))$ instead of $(\forall n, (\forall k < n, P(k)) \rightarrow P(n)) \rightarrow \forall n, P(n)$. This cost only 1 mark — even though it's a serious omission.

Part (c) [4 MARKS]

Explain how the principle of complete induction allows us to conclude that $P(1)$ is true unconditionally.

SAMPLE SOLUTION:

$P(0)$ is true because $(\forall k < 0, P(k)) \rightarrow P(0)$ is equivalent to $P(0)$ (since $\forall k < 0, P(k)$ is vacuously true).

$P(1)$ is true because $(\forall k < 1, P(k)) \rightarrow P(1)$ is equivalent to $P(0) \rightarrow P(1)$ and $P(0)$ is true.

MARKING SCHEME:

- **Structure:** [1 mark]
Clear attempt to argue that $P(1)$ holds by “unrolling” the inductive argument.
- **Correctness:** [3 marks]
Correct arguments that $P(0)$ holds and that $P(1)$ holds.

MARKER’S COMMENTS:

- This question was fairly poorly done. A lot of people wrote long-winded and confusing solutions that did nothing more than rephrase the principle of complete induction without actually answering the question.
What you need to **clearly** answer is why $P(0)$ is true unconditionally, and why this implies $P(1)$. (Marking Code: “P0”).
The marks lost varied depending on how long-winded and confusing the solution was. You could have lost as few as 1 mark or up to all marks if you were completely wrong. (Most people lost about 2 marks).

Question 2. [10 MARKS]

Fill in each proof structure below so that it is correct, *i.e.*, so that the conclusion follows from the proof.

Part (a) [4 MARKS]

Predicate: $P(n)$.

Base Case(s): Prove $P(1)$.

Ind. Hyp.: Suppose $n \geq 1$ and n is a power of 3 and $P(n)$.

Ind. Step: Prove $P(3n)$.

Conclusion: $\forall n \in \mathbb{N}, n$ is a power of 3 $\implies P(n)$.

SAMPLE SOLUTION: (See above.)

MARKING SCHEME:

- **Correctness:** [4 marks]
1 mark for each correct element, -1 mark for each incorrect element. Point out unnecessary elements (but don’t take marks off for them).

MARKER’S COMMENTS:

- Incorrect base case (*e.g.*, $P(0)$): -1 mark.
- In the IH, many people omitted the assumption that n is a power of 3: -1 mark. (Marking code: “NP3”).
- Some people tried proving $P(n + 1)$ in the IS or tried specifying $n + 1$ to be a power of 3: -1 mark.

Part (b) [3 MARKS]**Predicate:** $P(n)$.**Ind. Hyp.:** Suppose $n \geq 2$ and $\forall k \in \{2, \dots, n-1\}, P(k)$.**Ind. Step:** Prove $P(n)$.**Conclusion:** $\forall n \geq 2, P(n)$.

SAMPLE SOLUTION: (See above.)

MARKING SCHEME:

• **Correctness:** [3 marks]

1 mark for each correct element, -1 mark for each incorrect element. Point out unnecessary elements (but don't take marks off for them).

MARKER'S COMMENTS:

- Some people didn't specify the assumption that $n \geq 2$ at the beginning of the IH. That is, they would specify " $\forall k \in \{2, \dots, n-1\}$ assume $P(k)$ " (or some variant of that) but omitted saying what n was: -1 mark.
- Some people didn't use complete induction — *i.e.*, they simply said "Let $n \geq 2$, and assume $P(n)$ ". The way the proof structure is laid out, however, requires the use of complete induction: -1 mark. (Marking code: "FA".)

Part (c) [3 MARKS]**Predicate:** $P(n)$.**Base Case(s):** Proof of $P(0)$.**Ind. Hyp.:** Suppose $n \in \mathbb{Z}$ and $P(n)$.**Ind. Step:** Prove $P(n+1)$ and $P(n-1)$.**Conclusion:** $\forall n \in \mathbb{Z}, P(n)$.

SAMPLE SOLUTION: (See above.)

MARKING SCHEME:

• **Correctness:** [3 marks]

1 mark for each correct element, -1 mark for each incorrect element. Point out unnecessary elements (but don't take marks off for them).

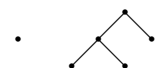
MARKER'S COMMENTS:

- Most people did well.
- Incorrect base case, or specifying \mathbb{N} instead of \mathbb{Z} in the conclusion: -1 mark.
- If there were other errors or confusing statements, more marks were taken off.

Question 3. [13 MARKS]

Recall that the *height* of a binary tree is defined as the maximum number of nodes on any path from the root of the tree to one of its leaves. For example, the first tree on the right (with a single node) has height 1 and the second tree has height 3 — the empty tree (with no node) has height 0.

Prove that for all $h \in \mathbb{N}$, every binary tree of height h contains at most $2^h - 1$ nodes. Keep in mind that your proof will be marked on its structure at least as much as on its content, so write it up carefully.



SAMPLE SOLUTION:

We prove the claim by induction on $h \geq 1$.

- **Predicate:** $P(h) =$ “Every binary tree of height h contains at most $2^h - 1$ nodes”.
- **Base Case:** If $h = 0$, the only tree of height 0 is the empty tree, and $0 \leq 1 - 1 = 2^0 - 1$, so $P(0)$ holds.
- **Ind. Hyp.:** Suppose $h > 0$ and $\forall k \in \{0, \dots, h - 1\}, P(k)$, *i.e.*, every binary tree of height k contains at most $2^k - 1$ nodes, for $k = 0, 1, \dots, h - 1$.
- **Ind. Step:** Suppose T is a tree of height h . Since $h > 0$, T is not empty and contains at least a root r . Let L be the subtree rooted at the left child of r and let R be the subtree rooted at the right child of r . The height of L is at most $h - 1$ and the height of R is at most $h - 1$ (by definition of height), so by the IH, L contains at most $2^{h-1} - 1$ nodes and R contains at most $2^{h-1} - 1$ nodes. But then, T contains at most $1 + 2^{h-1} - 1 + 2^{h-1} - 1 = 2^h - 1$ nodes. Since T was arbitrary, this holds for all binary trees of height h , *i.e.*, $P(h)$.
- **Conclusion:** For all $h \in \mathbb{N}$, every binary tree of height h contains at most $2^h - 1$ nodes.

MARKING SCHEME:

- **Structure:** [3 marks]
Clear attempt to give a correctly structured proof by induction, including defining the predicate, proving the base case(s), stating the inductive hypothesis, proving the inductive step, and stating the conclusion — independently of the correctness of any of those steps.
- **Predicate:** [1 marks]
Clear and correct definition of the predicate.
- **Base Case:** [2 marks]
Correct proof of base case, for correct value of h .
- **Ind. Hyp.:** [2 marks]
Clear and correct statement of inductive hypothesis, which “connects” properly with the base case and inductive step.
- **Ind. Step:** [5 marks]
Clear and correct proof of inductive step, including making appropriate use of the inductive hypothesis.

MARKER’S COMMENTS:

- Some people tried solving using simple induction. This will cause problems in the induction step. In the IS, students who tried solving things in this manner would consider a binary tree of height $h + 1$, and look at the left and right subtrees of the root node. They would then say the height of the left and right subtrees are h , apply the induction hypothesis, and reach the desired conclusion. Unfortunately, one of the subtrees may have height less than h , and unless you have used complete induction, you cannot conclude anything about that subtree in the IS: -1 mark from the IH, -1 mark from the IS. (Marking code: “CI”.)
- Some people had the right idea, but muddled the wording. 1 to 2.5 marks lost depending on how severe the problem was. (Marking code: “M”.)
- Some students assumed $h > 0$ and $P(h)$ in the IH, and then in the IS tried “adding” a node (or nodes) to the tree of height h in some way to construct a tree of height $h + 1$ so that $P(h + 1)$ was true. The problem with doing this is that in the IS, in order to conclude $P(h + 1)$, you must prove that EVERY tree of height $h + 1$ has at most $2^{h+1} - 1$ nodes, and the only way to do that is to START with an

arbitrary tree of height $h + 1$ – simply “adding” nodes to a tree of height h may only yield very specific types of trees of height $h + 1$: -2 or -3 marks depending on how the solution was worded. (Marking code: “ND”.)

- An alternative solution that I accepted went along the following lines. First prove by induction that $\forall h \in \mathbb{N}$, every “perfect” binary tree of height h has exactly $2^h - 1$ nodes — a binary tree if “perfect” when it is full (every internal node has two children) and all leaves are at the same depth. (This can be proved by simple induction). Then argue that any binary tree of height h cannot have more nodes than a complete binary tree of height h . (Note: To get marks for this solution, it had to be done cleanly. Some people tried making arguments about complete binary trees in the middle of the IS when they were trying to prove the more general conjecture about any binary tree. Such solutions received fewer marks.)
- In the IS, some people weren’t clear where they used the IH: -1 mark. (Marking code: “R”.)

Bonus. [6 MARKS]

WARNING! This question is difficult and credit will only be given for making *significant* progress toward a correct answer — in particular, you will NOT get 20% for writing “I don’t know”. Please attempt this only after you have completed the rest of the test.

Write a detailed proof structure that could be used to prove $\forall n \in \mathbb{Q}, P(n)$ (\mathbb{Q} is the set of rational numbers), and explain briefly why your proof structure is correct.

SAMPLE SOLUTION:

- **Base Case:** Prove $P(0)$.
- **Ind. Hyp.:** Suppose $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $b \neq 0$, $P(a)$, and $P(b)$.
- **Ind. Step:** Prove $P(a + 1)$, $P(a - 1)$, and $P(a/b)$.

Proving $P(a) \rightarrow P(a + 1)$ and $P(a) \rightarrow P(a - 1)$ starting with $a = 0$ allows us to conclude that $P(n)$ holds for all $n \in \mathbb{Z}$. Proving $P(a) \wedge P(b) \rightarrow P(a/b)$ starting with $a, b \in \mathbb{Z}$ ($b \neq 0$) allows us to conclude that $P(n)$ holds for all $n \in \mathbb{Q}$.

Another way to see this: $\forall p/q \in \mathbb{Q}$, $P(p/q)$ holds because $P(p)$ holds ($P(0) \rightarrow P(\pm 1) \rightarrow \dots \rightarrow P(p)$) and $P(q)$ holds ($P(0) \rightarrow P(\pm 1) \rightarrow \dots \rightarrow P(q)$), and $P(p) \wedge P(q) \rightarrow P(p/q)$.

MARKING SCHEME:

- **Format:** [1 mark]
Clear attempt to give a well-written proof structure together with a careful argument that the structure is correct — even if the proof structure is not, in fact, correct.
- **Structure:** [3 marks]
Correct proof structure.
- **Argument:** [2 marks]
Good argument of correctness.

MARKER’S COMMENTS:

- Very few people got marks. In particular, simply putting down “Base Case, Ind. Hyp., Ind. Step, Conclusion” with no content was worth 0.