Duration: **50 minutes**
Aids Allowed: **NONE** (in particular, no calculator)

**Student Number:** |_|_|_|_|_|_|_|_|_|

**Last (Family) Name(s):** ____________________

**First (Given) Name(s):** ____________________

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*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above,
and read the instructions below *carefully*.)

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This term test consists of 4 questions on 10 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, write your student number where indicated at the bottom of every page (except page 1), and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result or theorem covered in lectures, tutorials, assignments, or in the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

If you are unable to answer a question (or part of a question), you will get 20% of the marks for the question (or part of the question) if you state clearly that you do not know how to answer. Note that you will *not* get those marks if your answer contains contradictory statements (such as "I don't know" followed or preceded by parts of a solution that have not been crossed off).

MARKING GUIDE

# 1: ______/10

# 2: ______/ 8

# 3: ______/ 5

# 4: ______/ 7

BONUS MARKS: ______/ 2

TOTAL: ______/30

 *Good Luck!* 

*Use this page for rough work — clearly indicate any section(s) to be marked.*

## Question 1. [10 MARKS]

### Part (a) [2 MARKS]

Write the principle of **simple induction** for predicate $P(n)$, in symbolic form.

### Part (b) [8 MARKS]

Prove that $\forall n \in \mathbb{N}, f(n) = 2^{\lceil (n+1)/2 \rceil} - 2$, where $f(n) = \begin{cases} 0 & \text{if } n = 0 \text{ or } n = 1, \\ 2f(n-2) + 2 & \text{if } n > 1. \end{cases}$

(Write your proof carefully: marks will be awarded for the clarity and correctness of your proof structure.)

*Use this page for rough work — clearly indicate any section(s) to be marked.*

## Question 2. [8 marks]

Use induction to prove that for all **even** $n \geq 2$, every sequence of $n$ bits that starts and ends with the same value (both 0 or both 1) contains two consecutive bits with the same value. For example, the sequence of 6 bits 010100 contains two consecutive 0s in the last two bits.
(Write your proof carefully: marks will be awarded for the clarity and correctness of your proof structure.)

## Bonus. [2 marks]

Answer the question using *well ordering* instead of induction. If you attempt this, check the box below.

☐ Please mark my proof as a proof by well ordering (otherwise, it will be marked as a proof by induction).

*Use this page for rough work — clearly indicate any section(s) to be marked.*

## Question 3. [5 MARKS]

Perform repeated substitution on the following recurrence to find an approximate closed form for $T(n)$. Show all of your work. You do **not** have to prove anything for this question.

$$T(n) = \begin{cases} n & \text{if } n = 0 \text{ or } n = 1, \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + n^2 - 1 & \text{if } n > 1. \end{cases}$$

(You may use, without proof, the fact that for any real number $r \neq 1$, $1 + r + \cdots + r^{k-1} = \frac{1-r^k}{1-r} = \frac{r^k-1}{r-1}$.)

*Use this page for rough work — clearly indicate any section(s) to be marked.*

## Question 4. [7 MARKS]

For $n \geq 1$, let $P(n)$ be the number of ways to parenthesize the product of $n$ numbers $a_1 \cdot a_2 \cdots a_n$ so that each multiplication is applied to exactly two operands. For example, $P(4) = 5$ because there are exactly 5 ways to parenthesize the product $a_1 \cdot a_2 \cdot a_3 \cdot a_4$:
$(a_1)\big((a_2)\big((a_3)(a_4)\big)\big)$ $(a_1)\big(\big((a_2)(a_3)\big)(a_4)\big)$ $\big((a_1)(a_2)\big)\big((a_3)(a_4)\big)$ $\big((a_1)\big((a_2)(a_3)\big)\big)(a_4)$ $\big(\big((a_1)(a_2)\big)(a_3)\big)(a_4)$
Give a recurrence for $P(n)$, including appropriate base case(s), and justify that your recurrence applies (based on the meaning of $P(n)$). HINT: Think about the possible ways to put in the first two pairs of parentheses (to indicate the last multiplication to be performed).

**On this page, please write nothing except your name.**

**Last (Family) Name(s):** ______________________________

**First (Given) Name(s):** ______________________________