Worth: 8% **Due:** By 6pm on Wednesday 4 March

1. Simplify each sentence below so that it uses as few connectives and variables as possible—*without using truth tables*.

   (a) $(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r)$

   (b) $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

   (c) $p \wedge (q \vee p)$

2. Recall the following statements from Assignment 1:

$$(S_1) \quad \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge (a_j > a_i \vee a_j < a_i)$$

$$(S_2) \quad \forall i \in \mathbb{N}, \big(\exists j \in \mathbb{N}, j > i \wedge a_j > a_i\big) \wedge \big(\exists j \in \mathbb{N}, j > i \wedge a_j < a_i\big)$$

   And the following sequences:

$$(A_1) \quad 1,2,3,2,3,4,3,4,5,4,5,6,\ldots; a_n = \begin{cases} 1 + n/3 & \text{if } \exists k \in \mathbb{N}, n = 3k, \\ 2 + (n-1)/3 & \text{if } \exists k \in \mathbb{N}, n = 3k+1, \\ 3 + (n-2)/3 & \text{if } \exists k \in \mathbb{N}, n = 3k+2. \end{cases}$$

$$(A_2) \quad 10,9,8,7,6,5,4,3,2,1,1,1,\ldots; a_n = \begin{cases} 10 - n & \text{if } n \leqslant 9, \\ 1 & \text{if } n > 9. \end{cases}$$

   (a) Write a detailed structured proof that (S_1) is true for (A_1).

   (b) Write a detailed structured proof that (S_2) is false for (A_2).

3. Often in mathematics, we want to express the fact that there exists *exactly* one element $x \in D$ with a certain property $P(x)$ (sometimes expressed as "there is a unique $x \in D$ such that $P(x)$"). This is represented symbolically with the notation "$\exists! x \in D, P(x)$", a shorthand for the sentence

$$\exists x \in D, \forall y \in D, P(y) \Leftrightarrow y = x.$$

   Give a detailed structured proof that if a and b are distinct integers (*i.e.*, $a \neq b$) such that $a + b$ is even, then there is a unique integer c such that $|a - c| = |b - c|$.

4. For all real numbers x, define $\lfloor x \rfloor$ as in the lecture notes, and $\lceil x \rceil$ as the smallest integer greater than or equal to x.

   Note that for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the unique integer that satisfies $x - 1 < \lfloor x \rfloor \leqslant x$, and $\lceil x \rceil$ is the unique integer that satisfies $x \leqslant \lceil x \rceil < x + 1$.

   For each question below, write a detailed structured proof.

   (a) Prove or disprove that $\forall x \in \mathbb{R}, \lceil -x \rceil = -\lfloor x \rfloor$.

   (b) Prove or disprove that $\forall x \in \mathbb{R}, \forall k \in \mathbb{Z}, \lfloor x + k \rfloor = \lfloor x \rfloor + k$.

   (c) Prove or disprove that $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, \lfloor n \cdot x \rfloor = n \cdot \lfloor x \rfloor$.

5. Give a detailed structured proof that there is no rational number r such that $r^3 + r + 1 = 0$.

   (Hint: Use a proof by contradiction and introduce cases for even/odd.)

6. For all positive real numbers x, y:

   - the "arithmetic mean" of x and y is defined as $(x + y)/2$;
   - the "geometric mean" of x and y is defined as $\sqrt{xy}$;
   - the "harmonic mean" of x and y is defined as $2/\left(\frac{1}{x} + \frac{1}{y}\right)$.

   Formulate a conjecture about the relationship between all three of these values, and write a detailed structured proof of your conjecture—for full marks, your conjecture should be as strong as possible (*e.g.*, saying something simple like "all three means are positive" will be worth very few marks).