1. Justify each equivalence below by providing a derivation from one expression to the other (with a brief justification for each step of your derivation), or show that the equivalence does not hold (warning; you cannot use a derivation to show non-equivalence).

(a) \((P \Rightarrow Q) \land (P \Rightarrow R) \iff P \Rightarrow (Q \land R)\)

This equivalence holds.
\[
(P \Rightarrow Q) \land (P \Rightarrow R) \iff (-P \lor Q) \land (-P \lor R)
\]
\[
\iff \neg P \lor (Q \land R)
\]
\[
\iff P \Rightarrow (Q \land R)
\]

(b) \((P \Rightarrow R) \land (Q \Rightarrow R) \iff (P \land Q) \Rightarrow R\)

This equivalence does not hold. Suppose \(P\) is True, \(Q\) is False, and \(R\) is False. Then
\[
(P \Rightarrow R) \land (Q \Rightarrow R) = (True \Rightarrow False) \land (False \Rightarrow False)
\]
\[
= False \land True
\]
\[
= False
\]

but
\[
(P \land Q) \Rightarrow R = (True \land False) \Rightarrow False
\]
\[
= False \Rightarrow False
\]
\[
= True
\]

(c) \(P \iff Q \iff (P \land Q) \lor (-P \land \neg Q)\)

This equivalence holds.
\[
P \iff Q \iff (P \Rightarrow Q) \land (Q \Rightarrow P)
\]
\[
\iff (-P \lor Q) \land (-Q \lor P)
\]
\[
\iff (((-P \lor Q) \land \neg Q) \lor ((-P \lor Q) \land P))
\]
\[
\iff (-P \land \neg Q) \lor (Q \land \neg Q) \lor (-P \land P) \lor (Q \land P)
\]
\[
\iff (-P \land \neg Q) \lor (Q \land P)
\]
\[
\iff (P \land Q) \lor (-P \land \neg Q)
\]
\[
\text{(commutativity)}
\]
2. An “interpretation” for a logical statement consists of a domain of elements and a meaning for each predicate symbol. (When the statement contains no quantifiers or open variables, an interpretation consists simply of a truth value (True/False) for each predicate symbol.) For each statement below, provide an interpretation under which the statement is true and another interpretation under which the statement is false—if either case is not possible, clearly explain why.

(a) \( \forall x \in D, \forall y \in D, P(x, y) \)

Let \( D = \{1, 2\} \) and \( P(x, y) \): “\( x < y \)”. Then, \( \forall x \in D, \forall y \in D, P(x, y) \) is False because \( P(2, 1) \) is False.

Let \( D = \{1\} \) and \( P(x, y) \): “\( x = y \)”. Then \( \forall x \in D, \forall y \in D, P(x, y) \) is True because \( P(1, 1) \) is True.

(b) \((P \land Q) \Rightarrow (P \lor Q)\)

Let \( P = False \) and \( Q = True \). Then

\[
(P \land Q) \Rightarrow (P \lor Q) = (False \land True) \Rightarrow (False \lor True) = True \Rightarrow True = True
\]

It is impossible to make the statement False because this would require \( P \land Q \) to be True (meaning both \( P \) and \( Q \) are True) while at the same time \( P \lor Q \) is False (meaning both \( P \) and \( Q \) are False).

(c) \( \forall x \in D, \exists y \in D, P(x, y) \Rightarrow \exists y \in D, \forall x \in D, P(x, y) \)

Let \( D = \{1, 2\} \) and \( P(x, y) \): “\( x \) is an integer multiple of \( y \)”. Then, \( \forall x \in D, \exists y \in D, P(x, y) \) is True because \( P(1, 1) \) and \( P(2, 2) \) are both True (i.e., we can always pick \( y = x \) for every \( x \)). Moreover, \( \exists y \in D, \forall x \in D, P(x, y) \) is True because \( P(1, 1) \) and \( P(2, 1) \) are both True (i.e., \( y = 1 \) works for every \( x \)). Hence, the entire statement is True.

Let \( D = \{1, 2\} \) and \( P(x, y) \): “\( x \neq y \)”. Then, \( \forall x \in D, \exists y \in D, P(x, y) \) is False because \( P(1, 2) \) and \( P(2, 1) \) are both True (i.e., we can always pick \( y \) to be different from \( x \)). However, \( \exists y \in D, \forall x \in D, P(x, y) \) is False because there is no value of \( y \in D \) that is different from every other element of \( D \). Hence, the entire statement is False.