Duration: **75 minutes**
Aids Allowed: **none**

Student Number: |_|_|_|_|_|_|_|_|_|

Family Name(s): ______________________________

Given Name(s): ______________________________

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*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This term test consists of 3 questions on 8 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, write your student number where indicated at the bottom of every odd-numbered page (except page 1), and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result or theorem covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do — part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

If you are unable to answer a question (or part), you will get 20% of the marks for that question (or part) if you write "I don't know" and nothing else — you will *not* get those marks if your answer is completely blank, or if it contains contradictory statements (such as "I don't know" followed or preceded by parts of a solution that have not been crossed off).

MARKING GUIDE

1: ______/18

2: ______/18

3: ______/12

TOTAL: ______/48

Use this page for rough work — clearly indicate any section(s) to be marked.

Question 1. [18 MARKS]

In Assignment 2, we defined $\exists! x \in D, P(x)$ as a shorthand for $\exists x \in D, \forall y \in D, P(y) \Leftrightarrow y = x$. In this question, you will show that this definition is equivalent to the more "traditional" representation $\exists x \in D, P(x) \wedge \forall y \in D, y \neq x \Rightarrow \neg P(y)$.

Part (a) [12 MARKS]

Prove that for any $x \in D$, $P(x) \Leftrightarrow \forall y \in D, y = x \Rightarrow P(y)$. (HINT: Use proof structures.)

Part (b) [6 MARKS]

For this part, you may refer to any equivalence from the following list by its number.

(1) $(p \Leftrightarrow q) \iff (p \Rightarrow q) \wedge (q \Rightarrow p)$ (2) $p \Rightarrow q \iff \neg q \Rightarrow \neg p$

(3) $(\forall x \in D, p(x) \wedge q(x)) \iff (\forall x \in D, p(x)) \wedge (\forall x \in D, q(x))$ (4) $p \wedge q \iff q \wedge p$

You may also make use of the result from part (a), even if you did not answer that part.

Prove that for any $x \in D$, $(\forall y \in D, P(y) \Leftrightarrow y = x) \iff (P(x) \wedge \forall y \in D, y \neq x \Rightarrow \neg P(y))$.

Use this page for rough work — clearly indicate any section(s) to be marked.

Question 2. [18 MARKS]

Write a detailed structured proof that $\forall i_0 \in \mathbb{N}, \exists i_1 \in \mathbb{N}, \exists i_2 \in \mathbb{N}, i_2 \neq i_1 \wedge i_1 \neq i_0 \wedge i_0 \neq i_2 \wedge a_{i_2} = a_{i_1} = a_{i_0}$ for the sequence defined by $a_n = \lfloor n/3 \rfloor$ (*i.e.*, $0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, \ldots$).
(HINT: Use the following fact: $(*)$ $\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, n = 3k \vee n = 3k + 1 \vee n = 3k + 2$.)

Use this page for rough work — clearly indicate any section(s) to be marked.

Question 3. [12 marks]

Describe two different strategies/approaches you could use to attempt to solve the following problem, and explain what you hope to obtain from each strategy.

Imagine that you have five integers arranged in a circle. The integers can be arbitrary but their sum must be positive. An "atomic action" consists in choosing a negative integer (if there is one), adding its value to both of its immediate neighbours, and taking the absolute value of the negative integer chosen. For example, if we start with the values on the left, we reach the values in the middle by performing an atomic action on -4, and the values on the right by following this with an atomic action on -7.

1	-3	-10
-7 -4	-7 4	7 4
5 8	5 4	-2 4

Prove that no matter what integers we start with (as long as their sum is positive), if we perform repeated atomic actions, eventually, all of the values will become non-negative.

On this page, please write nothing except your name.

Family Name(s): ______________________________

Given Name(s): ______________________________