**Worth:** 8% **Due:** By 6pm on Wednesday 28 January

[20] 1. Let $U$ denote the set of all faculty members and students at the University of Toronto, along with the students' parents. Let Alice and Bruno be specific individuals in $U$. Consider the following predicates, along with their meaning: $F(x)$: "$x$ is a faculty member"; $S(x)$: "$x$ is a student"; $M(x)$: "$x$ attended/will attend the meeting"; $K(x, y)$: "$x$ knows $y$"; $P(x, y)$: "$x$ is a parent of $y$".

Using only the domain, constants, and predicates above (in addition to appropriate connectives and quantifiers), translate each sentence below, *i.e.*, give a natural English sentence that corresponds to each symbolic sentence, and give a clear symbolic sentence that corresponds to each English sentence. State clearly any assumptions you might need to make.

(a) If all students attended the meeting, then all faculty members attended the meeting.
(b) $\exists x \in U, \exists y \in U, S(x) \wedge P(y, x) \wedge \neg M(y)$
(c) Some student knows everyone who attended the meeting.
(d) $\forall x \in U, S(x) \wedge K(\mathsf{Bruno}, x) \Rightarrow \neg M(x)$
(e) No student who attended the meeting knows a faculty member.
(f) $\forall x \in U, F(x) \wedge M(x) \Rightarrow K(\mathsf{Bruno}, x)$
(g) Alice and Bruno attended the meeting, but neither of them knows any student.
(h) $\forall x \in U, \forall y \in U, (\exists z \in U, \exists w \in U, S(z) \wedge M(z) \wedge P(x, z) \wedge S(w) \wedge M(w) \wedge P(y, w)) \Rightarrow K(x, y)$
(i) Alice will attend the meeting, unless she knows no student who will attend the meeting.
(j) $\exists x \in U, F(x) \wedge M(x) \wedge K(x, \mathsf{Alice}) \wedge \forall y \in U, P(y, \mathsf{Alice}) \Rightarrow K(x, y)$

[15] 2. Consider the following sentence about integers $a, b, c$:

If $a$ divides $bc$, then $a$ divides $b$ or $a$ divides $c$.

For each sentence below, state whether it is the negation of, the converse of, the contrapositive of, unrelated to, or equivalent to the sentence above. Justify each of your answers briefly (*e.g.*, by writing both sentences in symbolic notation).

(a) If $a$ divides $b$ or $a$ divides $c$, then $a$ divides $bc$.
(b) If $a$ does not divide $b$ or $a$ does not divide $c$, then $a$ does not divide $bc$.
(c) $a$ divides $bc$ and $a$ does not divide $b$ and $a$ does not divide $c$.
(d) If $a$ does not divide $b$ and $a$ does not divide $c$, then $a$ does not divide $bc$.
(e) If $a$ divides $bc$ and $a$ does not divide $c$, then $a$ divides $b$.

[15] 3. Consider the following statements about sequences of natural numbers $a_0, a_1, a_2, \ldots$—(in this course, the natural numbers start at 0, *i.e.*, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$):

$$(S_1) \quad \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge (a_j > a_i \vee a_j < a_i)$$

$$(S_2) \quad \forall i \in \mathbb{N}, \big(\exists j \in \mathbb{N}, j > i \wedge a_j > a_i\big) \wedge \big(\exists j \in \mathbb{N}, j > i \wedge a_j < a_i\big)$$

And the following sequences:

$$(A_1) \quad 1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 5, 6, \ldots$$

$$(A_2) \quad 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 1, 1, \ldots$$

For each sequence and each statement, state whether the statement is true or false for the sequence and justify briefly.

[10] 4. Find three sets $A, B, C$ *with as few elements as possible* so that statement $(S_3)$ below is true but statement $(S_4)$ is false, and justify briefly that this is the case.

$$(S_3) \quad \forall x \in A, \exists y \in B, x + y \in C$$

$$(S_4) \quad \exists y \in B, \forall x \in A, x + y \in C$$

[20] 5. At a trial, four witnesses give the following testimony.

**Alice:** Either Bruno is guilty and Carol innocent, or Bruno is innocent and Carol guilty.

**Bruno:** If Alice or Danny is innocent, then so is Carol.

**Carol:** Either Alice or Bruno is guilty, but I am innocent.

**Danny:** Bruno is innocent or Carol is guilty if and only if Bruno is innocent and Carol is guilty.

(a) Is it possible that everyone is telling the truth? (Assume that each person is either innocent or guilty.) Explain why or why not.

(b) If each innocent person always tells the truth, and each guilty person always lies, is it possible to determine who is guilty and who is innocent? If so, how?